

# SLAVNOV DETERMINANTS, YANG-MILLS STRUCTURE CONSTANTS, AND DISCRETE KP

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**ABSTRACT.** Using Slavnov's scalar product of a Bethe eigenstate and a generic state in closed XXZ spin- $\frac{1}{2}$  chains, with possibly twisted boundary conditions, we obtain determinant expressions for tree-level structure constants in 1-loop conformally-invariant sectors in various planar (super) Yang-Mills theories. When certain rapidity variables are allowed to be free rather than satisfy Bethe equations, these determinants become discrete KP  $\tau$ -functions.

*Dedicated to Professor M Jimbo on his 60th birthday.*

## 0. OVERVIEW

Classical integrable models, in the sense of integrable hierarchies of nonlinear partial differential equations that admit soliton solutions, and quantum integrable models, in the sense of Yang-Baxter integrability, are topics that Prof M Jimbo continues to make profound contributions to since more than three decades.

They are also topics that, since the late 1980's, have made increasingly frequent contacts with, and have lead to definite advances in modern quantum field theory. Amongst the most important of these contacts are discoveries of integrable structures on both sides of Maldacena's conjectured AdS/CFT correspondence [1]. From 2002 onward, classical integrability was discovered in free superstrings<sup>1</sup> on the AdS side of AdS/CFT [2, 3], and quantum integrability in the planar limit<sup>2</sup> of  $\mathcal{N} = 4$  supersymmetric Yang-Mills on the CFT side [4, 5, 6]. Further, examples of integrability that are restricted 1-loop level were discovered in planar Yang-Mills theories with fewer supersymmetries and in QCD [7, 8]. In the sequel, we use YM for Yang-Mills theories in general, and  $\text{SYM}_{\mathcal{N}}$  for  $\mathcal{N}$ -extended supersymmetric Yang-Mills.

**0.1. Scope of this work.** In this work, we restrict our attention to quantum field theories that are **1.** planar, so that the methods of integrability have a chance to work, **2.** weakly-coupled, so that perturbation theory makes sense and we can focus our attention to 1-loop level, and **3.** conformally-invariant at 1-loop level, so they allow an exact mapping to Heisenberg spin-chains, that is spin-chains with nearest neighbour interactions that can be solved using the algebraic Bethe Ansatz. In the sequel, we consider only Heisenberg spin- $\frac{1}{2}$  chains.

Even within the above restrictions, our subject is still very broad and we can only review the basics needed to obtain our results. For an introduction to the vast subject of integrability in AdS/CFT, we refer to [9] and references therein<sup>3</sup>.

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*Key words and phrases.* Yang-Mills theories. Heisenberg spin chain. Six-vertex model.

<sup>1</sup>Superstrings with tree-level interactions only, and no spacetime loops.

<sup>2</sup>The limit in which the number of colours  $N_c \rightarrow \infty$ , the gauge coupling  $g_{YM} \rightarrow 0$ , while the 't Hooft coupling  $\lambda = g_{YM}^2 N_c$  remains finite.

<sup>3</sup>Further highlights of integrability in modern quantum field theory and in string theory include **1.** Classical integrable hierarchies in matrix models of non-critical strings, from the late 1980's [10], **2.** Finite gap solutions in Seiberg-Witten theory of low-energy  $\text{SYM}_2$  in the mid 1990's [11, 12, 13], **3.** Integrability in QCD scattering amplitudes in the mid 1990's [14, 8], **4.** Free fermion methods in works of Nekrasov, Okounkov, Nakatsu, Takasaki and others on Seiberg-Witten theory, in the 2000's [15, 16], **5.** Integrable spin chains in works of Nekrasov, Shatashvili and others on  $\text{SYM}_2$ , in the 2000's [17], **6.** Integrable structures, particularly the Yangian, that appear in recent studies of  $\text{SYM}_4$  scattering amplitudes [18]. There are many more.

**0.2. Conformal invariance and 2-point functions.** Any 1-loop conformally-invariant quantum field theory contains (up to 1-loop order) a basis of local scalar primary conformal composite operators <sup>4</sup>  $\{\mathcal{O}\}$  such that the 2-point functions can be written as

$$(1) \quad \langle \mathcal{O}_i(x) \overline{\mathcal{O}}_j(y) \rangle = \delta_{ij} \mathcal{N}_i |x - y|^{-2\Delta_i}$$

where  $\overline{\mathcal{O}}_i$  is the Wick conjugate of  $\mathcal{O}_i$ ,  $\Delta_i$  is the conformal dimension of  $\mathcal{O}_i$  and  $\mathcal{N}_i$  is a normalization factor. Later, we choose  $\mathcal{N}_i$  to be (the square root of) the Gaudin norm of the corresponding spin-chain state.

The primary goal of studies of integrability on the CFT side of AdS/CFT in the past ten years has arguably been the calculation of the spectrum of conformal dimensions  $\{\Delta_{\mathcal{O}}\}$  of local composite operators  $\{\mathcal{O}\}$ , and matching them with corresponding results from the strong coupling AdS side of AdS/CFT. This goal has by and large been achieved [9], and the next logical step is to study 3-point functions and their structure constants [19, 20, 21].

**0.3. 3-point functions and structure constants.** The 3-point function of three basis local operators such as those that appear in (1) is restricted (up to 1-loop order) by conformal symmetry to be of the form

$$(2) \quad \langle \mathcal{O}_i(x_i) \mathcal{O}_j(x_j) \mathcal{O}_k(x_k) \rangle = \left( \mathcal{N}_i \mathcal{N}_j \mathcal{N}_k \right)^{1/2} \frac{C_{ijk}}{|x_{ij}|^{\Delta_i + \Delta_j - \Delta_k} |x_{jk}|^{\Delta_j + \Delta_k - \Delta_i} |x_{ki}|^{\Delta_k + \Delta_i - \Delta_j}}$$

where  $x_{ij} = x_i - x_j$ , and  $C_{ijk}$  are structure constants. The structure constants  $C_{ijk}$  are the subject of this work. In [20], Escobedo, Gromov, Sever and Vieira (EGSV) obtained sum expressions for the structure constants of non-extremal single-trace operators in the scalar sector of SYM<sub>4</sub>. In [21], the sum expressions of EGSV were evaluated, and determinant expressions for the same structure constants were obtained <sup>5</sup>.

**0.4. Aims of this work.** We extend the results of [21] to a number of YM theories that are conformally invariant at least up to 1-loop level. We also show that the determinants that we obtain are discrete KP  $\tau$ -functions.

More precisely, **1.** We recall, and make explicit, a generalization of the restricted Slavnov scalar product used in [21] to twisted, closed and homogeneous XXZ spin- $\frac{1}{2}$  chains. That is, we allow for an anisotropy parameter  $\Delta \neq 1$ , as well as a twist parameter  $\theta \neq 0$  in the boundary conditions. The result is still a determinant. We use this result to obtain determinant expressions for the YM theories listed in subsection **0.5** <sup>6</sup>. **2.** Allowing certain rapidity variables in the determinant expressions to be free, rather than satisfy Bethe equations, we show that these rapidities can be regarded as Miwa variables. In terms of these Miwa variables, the determinants satisfy Hirota-Miwa equations and become discrete KP  $\tau$ -functions. The structure constants are recovered by requiring that the free variables are rapidities that label a gauge-invariant composite operator and satisfy Bethe equations.

**0.5. Type-A and Type-B YM theories.** We consider six planar, weakly-coupled YM theories. **1.** SYM<sub>4</sub> [22, 23], **2.** SYM<sub>4</sub><sup>M</sup>, which is an order- $M$  Abelian orbifold of SYM<sub>4</sub> that is  $\mathcal{N} = 2$  supersymmetric [24, 25], and **3.** SYM<sub>4</sub> <sup>$\beta$</sup> , which is a Leigh-Strassler marginal real- $\beta$  deformation of SYM<sub>4</sub> that is  $\mathcal{N} = 1$  supersymmetric [27, 25]. **4.** The complex scalar sector of pure SYM<sub>2</sub> [28, 7], **5.** The gluino sector of pure SYM<sub>1</sub> [7], and **6.** The gauge sector of QCD [7, 8].

<sup>4</sup>In this work, we restrict our attention to this class of local composite operators. In particular, we do not consider descendants or operators with non-zero spin, for which the 2-point and 3-point functions are different.

<sup>5</sup>Three operators  $\mathcal{O}_i$ , of length  $L_i$ ,  $i \in \{1, 2, 3\}$ , are non-extremal if  $l_{ij} = L_i + L_j - L_k > 0$ .

<sup>6</sup>The SYM<sub>4</sub> expression of [21] is a special case of the general expression obtained here.

These six theories are naturally divisible into two types. Type-**A** contains theories **1**, **2** and **3**, which are conformally-invariant to all orders in perturbation theory. Type-**B** contains theories **4**, **5** and **6**, which are conformally-invariant to 1-loop level only <sup>7</sup>.

Conformal invariance at 1-loop level, which is the case in all theories that we consider, is necessary and sufficient for our purposes because the mapping to spin- $\frac{1}{2}$  chains with nearest neighbour interactions breaks down at higher loops. Our results are valid only up to 1-loop level.

**0.6. Non-extremal operators.** In [20, 21], structure constants of three operators  $\mathcal{O}_i$  of length  $L_i$ ,  $i \in \{1, 2, 3\}$  were considered, and the condition that the operators are non-extremal, that is  $l_{ij} = L_i + L_j - L_k > 0$ , for all distinct  $i, j$  and  $k$ , was emphasized. The reason is that, in these works, one wished to compute the structure constants of three non-BPS operators. Using the analysis presented in this work, one can show that this requires the condition  $l_{ij} > 0$ . One can of course consider the special case where one of these parameters  $l_{ij} = 0$ , but then at least one of the three operators has to be BPS.

In type-**A** theories, which include  $\text{SYM}_4$ , we can compute non-trivial structure constants of three non-BPS operators, so we do that, and the condition  $l_{ij} > 0$  is satisfied. The case where one of these parameters vanishes, for example  $l_{23} = L_2 + L_3 - L_1 = 0$ , is allowed, but then either  $\mathcal{O}_2$  or  $\mathcal{O}_3$  has to be BPS. In type-**B** theories, we find that one of the three operators, which we choose to be  $\mathcal{O}_3$ , has to be BPS, hence the condition  $l_{ij} > 0$  is no longer significant and we consider operators such that  $l_{23} = L_2 + L_3 - L_1 = 0$ .

**0.7.  $SU(2)$  sectors that map to spin- $\frac{1}{2}$  chains.** We will not list the full set of fundamental fields in the gauge theories that we consider, but only those fundamental fields that form  $SU(2)$  doublets that map to states in spin- $\frac{1}{2}$  chains. All fields are in the adjoint of  $SU(N_c)$  and can be represented in terms of  $N_c \times N_c$  matrices.

- 1.**  $\text{SYM}_4$  contains six real scalars that form three complex scalars  $\{X, Y, Z\}$ , and their charge conjugates  $\{\bar{X}, \bar{Y}, \bar{Z}\}$ . Any pair of non-charge-conjugate scalars, *e.g.*  $\{Z, X\}$ , or  $\{Z, \bar{X}\}$ , forms a doublet that maps to a state in a closed periodic XXX spin- $\frac{1}{2}$  chain <sup>8</sup> [4, 23].
- 2.**  $\text{SYM}_4^M$  has the same fundamental charged scalar fields  $\{X, Y, Z\}$  and their charge conjugates, as  $\text{SYM}_4$ , so the same scalars form  $SU(2)$  doublets. Due to the orbifolding of the  $SU(2)$  sectors by the action of the discrete group  $\Gamma_M$ , these doublets map to states in a closed twisted XXX spin- $\frac{1}{2}$  chain. The twist parameter is a (real) phase  $\theta = \frac{2\pi}{M}$  [25].
- 3.**  $\text{SYM}_4^\beta$  has the same fundamental charged scalar fields  $\{X, Y, Z\}$  and their charge conjugates, as  $\text{SYM}_4$ , so the same scalars form  $SU(2)$  doublets. Due to the real- $\beta$  deformation, these doublets map to states in a closed twisted XXX spin- $\frac{1}{2}$  chain. The twist parameter is a (real) phase  $\theta = \beta$ , where  $\beta$  is the deformation parameter. [26, 25].
- 4.**  $\text{SYM}_2$  has a gluino field  $\lambda$  and its conjugate  $\bar{\lambda}$  that form a doublet that maps to a state in a closed untwisted XXZ spin- $\frac{1}{2}$  chain with  $\Delta = 3$  [28, 7].
- 5.**  $\text{SYM}_1$  has a complex scalar  $\phi$  and its conjugate  $\bar{\phi}$  that form a doublet that maps to a state in a closed untwisted XXZ spin- $\frac{1}{2}$  chain with  $\Delta = \frac{1}{2}$  [7].
- 6.** Pure QCD has light-cone derivatives  $\{\partial_+ A, \partial_+ \bar{A}\}$ , where  $A$  and  $\bar{A}$  are the transverse components of the gauge field  $A_\mu$ , that form a doublet that maps to a state in a closed untwisted XXZ spin- $\frac{1}{2}$  chain with  $\Delta = -\frac{11}{3}$  [7].

**0.8. Remark.** Theories **1**, **2** and **3**, that are conformally invariant to all orders, contain three charged scalars and their conjugates. These combine into various  $SU(2)$  doublets. Theories **4**,

<sup>7</sup>There are definitely more gauge theories that are conformally-invariant at 1-loop or more, with  $SU(2)$  sectors that map to states in spin- $\frac{1}{2}$  chains. Here we consider only samples of theories with different supersymmetries and operator content.

<sup>8</sup>XXX spin- $\frac{1}{2}$  chains are XXZ spin- $\frac{1}{2}$  chains with an anisotropy parameter  $\Delta = 1$ .

**5** and **6**, on the other hand, contain only one doublet. This fact affects the type of structure constants that we can compute in determinant form in Section 4 and 5<sup>9</sup>.

**0.9. Outline of contents.** In Section 1, we recall basic background information related to integrability in weakly coupled YM. In Section 2, we review standard facts on closed XXZ spin- $\frac{1}{2}$  chains with twisted boundary conditions. In particular, following [32], we introduce restricted versions  $S[L, N_1, N_2]$  of Slavnov's scalar product, that can be evaluated in determinant form<sup>10</sup>.

In Section 3, we review standard facts on the trigonometric six-vertex model, which is regarded as another way to view XXZ spin- $\frac{1}{2}$  chains in terms of diagrams that are convenient for our purposes. Following [33], we introduce the  $[L, N_1, N_2]$ -configurations that are central to our result. The determinant  $S[L, N_1, N_2]$ , obtained in Section 2, turns out to be the partition function of these  $[L, N_1, N_2]$ -configurations.

In Section 4, we recall the EGSV formulation of the structure constants of three non-extremal composite operators in the scalar sector of  $\text{SYM}_4$ . Since all Type-**A** theories, which include  $\text{SYM}_4$  and two other theories that are closely related to it, share the same set of fundamental charged scalar fields, namely  $\{X, Y, Z\}$  and their charge conjugates  $\{\bar{X}, \bar{Y}, \bar{Z}\}$ , our discussion applies to all of them in one go. Since the composite operators that we are interested in map to states in (generally twisted) XXX spin- $\frac{1}{2}$  chains, we express these structure functions in terms of rational six-vertex model configurations, and obtain determinant expressions for them.

In Section 5, we extend the above discussion to Type-**B** theories, which contain theories with only one  $SU(2)$  doublet that we can work with. Since the composite operators that we are interested in map to states in periodic XXZ spin- $\frac{1}{2}$  chains, we express these structure functions in terms of trigonometric six-vertex model configurations. We find that our method applies only when one of the operators is BPS-like (a single-trace of a power of one type of fundamental fields). We obtain determinant expressions for these objects, and find that the result is identical to that in type-**A**, apart from the fact that one of the operators is BPS-like.

In Section 6, we show that the determinant expressions are solutions of Hirota-Miwa equations, and thereby  $\tau$ -functions of the discrete KP hierarchy. In Section 7, we summarize our results.

## 1. BACKGROUND

Let us recall basic facts on integrability on the CFT side of AdS/CFT.

**1.1. Integrability in AdS/CFT.** In its strongest sense, the anti-de Sitter/conformal field theory (AdS/CFT) correspondence is the postulate that all physics, including gravity, in an anti-de Sitter space can be reproduced in terms of a conformal field theory that lives on the boundary of that space [30]. The first and most thoroughly studied example of the correspondence is Maldacena's original proposal that type-IIB superstring theory in an  $\text{AdS}^5 \times S^5$  geometry is equivalent to planar  $\text{SYM}_4$  on the 4-dimensional boundary of  $\text{AdS}^5$  [1].

Since its proposal in 1997, the AdS/CFT correspondence has passed every single check that it was subject to, and there was a large number of these. However, because the correspondence typically identifies one theory in a regime that is easy to study (for example, a weakly-coupled planar quantum field theory) to another theory in a regime that is hard to study (for example, a quantum free superstring theory in a strongly curved geometry), it has so far not been possible to prove it [9].

**1.2. The dilatation operator.** The generators of the conformal group in 4-dimensional space-time,  $SO(4, 2)$ , contain a dilatation operator  $D$  [31]. Every gauge-invariant operator  $\mathcal{O}$  in a YM theory, that is 1-loop conformally-invariant, is an eigenstate of  $D$  to that order in perturbation

<sup>9</sup>The fact that the structure constants in these two types of theories should be handled differently was pointed out to us by C Ahn and R Nepomechie.

<sup>10</sup>In [21],  $S[L, N_1, N_2]$  was denoted by  $S[L, \{N\}]$ .

theory. The corresponding eigenvalue  $\Delta_{\mathcal{O}}$ , which is the conformal dimension of  $\mathcal{O}$ , is the analogue of mass in massive, non-conformal theories.

**1.3. SYM<sub>4</sub> and spin chains. 1-loop results.** An  $SU(2)$  doublet of fundamental fields  $\{u, d\}$ , which could be any of those discussed in Subsection 0.7 above, is analogous to the  $\{\uparrow, \downarrow\}$  states of a spin variable on a single site in a spin- $\frac{1}{2}$  chain. Furthermore, the local gauge-invariant operators formed by taking single traces of a product of an arbitrary combination of  $u$  and  $d$  fields, such as  $\text{Tr}[uududduu \cdots uu]$ , is analogous to a state in a closed spin- $\frac{1}{2}$  chain.

In [4], Minahan and Zarembo made the above intuitive analogies exact correspondences by showing that the action of the 1-loop dilatation operator on single-trace operators in the  $SU(2)$  scalar subsector of SYM<sub>4</sub> is identical to the action of the nearest-neighbour Hamiltonian on the states in a closed periodic XXX spin- $\frac{1}{2}$  chain<sup>11</sup>. In this mapping, valid up to 1-loop level<sup>12</sup> single-trace operators with well-defined conformal dimensions map to eigenstates of the XXX Hamiltonian. The corresponding eigenvalues are the conformal dimensions  $\Delta_{\mathcal{O}}$ .

The above brief outline is all we need for the purposes of this work. For an in-depth overview, we refer the reader to [9].

## 2. THE XXZ SPIN- $\frac{1}{2}$ CHAIN

In this section, we recall basic facts related to the XXZ spin- $\frac{1}{2}$  chain that are needed in later sections. The presentation closely follows that in [33, 21], but adapted to closed XXZ spin chains with twisted boundary conditions.

**2.1. 1-dimensional lattice segments and spin variables.** Consider a length- $L$  1-dimensional lattice, and label the sites with  $i \in \{1, 2, \dots, L\}$ . Assign site  $i$  a 2-dimensional vector space  $h_i$  with the basis

$$(3) \quad |\wedge\rangle_i = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_i, \quad |\vee\rangle_i = \begin{pmatrix} 0 \\ 1 \end{pmatrix}_i$$

which we refer to as ‘up’ and ‘down’ states, and a spin variable  $s_i$  which can be equal to either of these states. The space of states  $\mathcal{H}$  is the tensor product  $\mathcal{H} = h_1 \otimes \cdots \otimes h_L$ . Every state in  $\mathcal{H}$  is an assignment  $\{s_1, s_2, \dots, s_L\}$  of  $L$  definite-value (either up or down) spin variables to the sites of the spin chain. In computing scalar products, as we do shortly, we think of states in  $\mathcal{H}$  as initial states.

**2.2. Initial spin-up and spin-down reference states.**  $\mathcal{H}$  contains two distinguished states,

$$(4) \quad |L^\wedge\rangle = \bigotimes_{i=1}^L \begin{pmatrix} 1 \\ 0 \end{pmatrix}_i, \quad |L^\vee\rangle = \bigotimes_{i=1}^L \begin{pmatrix} 0 \\ 1 \end{pmatrix}_i$$

where  $L^\wedge$  indicates  $L$  spin states that are all up, and  $L^\vee$  indicates  $L$  spin states that are all down. These are the initial spin-up and spin-down reference states, respectively.

<sup>11</sup>Minahan and Zarembo obtained their remarkable result in the context of the complete scalar sector of SYM<sub>4</sub>. The relevant spin chain in that case is  $SO(6)$  symmetric. Here we focus our attention on the restriction of their result to the  $SU(2)$  scalar subsector.

<sup>12</sup>We are interested in local single-trace composite operators that consist of many fundamental fields. These fields are interacting. In a weakly-interacting quantum field theory, one can consistently choose to ignore all interactions beyond a chosen order in perturbation theory. In the planar theory under consideration, perturbation theory can be arranged according to the number of loops in Feynman diagrams computed. In a 1-loop approximation, one keeps only 1-loop diagrams.

**2.3. Final spin-up and spin-down reference states, and a variation.** Consider a length- $L$  spin chain, and assign each site  $i$  a 2-dimensional vector space  $h_i^*$  with the basis

$$(5) \quad {}_i\langle\wedge| = \begin{pmatrix} 1 & 0 \end{pmatrix}_i, \quad {}_i\langle\vee| = \begin{pmatrix} 0 & 1 \end{pmatrix}_i$$

We construct a final space of states as the tensor product  $\mathcal{H}^* = h_1^* \otimes \cdots \otimes h_L^*$ .  $\mathcal{H}^*$  contains two distinguished states

$$(6) \quad \langle L^\wedge| = \bigotimes_{i=1}^L \begin{pmatrix} 1 & 0 \end{pmatrix}_i, \quad \langle L^\vee| = \bigotimes_{i=1}^L \begin{pmatrix} 0 & 1 \end{pmatrix}_i$$

where all spins are up, and all spins are down. These are the final spin-up and spin-down reference states. respectively. Finally, we consider the variation

$$(7) \quad \langle N_3^\vee, (L - N_3)^\wedge| = \bigotimes_{1 \leq i \leq N_3} \begin{pmatrix} 0 & 1 \end{pmatrix}_i \bigotimes_{(N_3+1) \leq i \leq L} \begin{pmatrix} 1 & 0 \end{pmatrix}_i$$

where the first  $N_3$  spins from the left are down, and all remaining spins are up.

**2.4. Pauli matrices.** We define the Pauli matrices

$$(8) \quad \sigma_m^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_m, \quad \sigma_m^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}_m, \quad \sigma_m^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_m$$

with  $i = \sqrt{-1}$ , and the spin raising/lowering matrices

$$(9) \quad \sigma_m^+ = \frac{1}{2}(\sigma_m^x + i\sigma_m^y) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}_m, \quad \sigma_m^- = \frac{1}{2}(\sigma_m^x - i\sigma_m^y) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}_m$$

where in all cases the subscript  $m$  is used to indicate that the matrices act in the vector space  $h_m$ .

**2.5. The Hamiltonian  $H$ .** The Hamiltonian of the finite length XXZ spin- $\frac{1}{2}$  chain is given by the equivalent expressions

$$(10) \quad H = \frac{1}{2} \sum_{m=1}^L \left( \sigma_m^x \sigma_{m+1}^x + \sigma_m^y \sigma_{m+1}^y + \Delta(\sigma_m^z \sigma_{m+1}^z - 1) \right) \\ = \sum_{m=1}^L \left( \sigma_m^+ \sigma_{m+1}^- + \sigma_m^- \sigma_{m+1}^+ + \frac{\Delta}{2}(\sigma_m^z \sigma_{m+1}^z - 1) \right)$$

where  $\Delta$  is the anisotropy parameter of the model, and where we assume the ‘twisted’ periodicity conditions

$$(11) \quad \sigma_{L+1}^\pm = e^{\pm i\theta} \sigma_1^\pm, \quad \sigma_{L+1}^z = \sigma_1^z$$

**2.6. The  $R$ -matrix.** From an initial reference state, we can generate all other states in  $\mathcal{H}$  using operators that flip the spin variables, one spin at a time. Defining these operators requires defining a sequence of objects. **1.** The  $R$ -matrix, **2.** The  $L$ -matrix, and finally, **3.** The monodromy or  $M$ -matrix.

The  $R$ -matrix is an element of  $\text{End}(h_a \otimes h_b)$ , where  $h_a, h_b$  are two 2-dimensional auxiliary vector spaces. The variables  $u_a, u_b$  are the corresponding rapidity variables. The  $R$ -matrix intertwines these spaces, and it has the  $(4 \times 4)$  structure

$$(12) \quad R_{ab}(u_a, u_b) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b[u_a, u_b] & c[u_a, u_b] & 0 \\ 0 & c[u_a, u_b] & b[u_a, u_b] & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{ab}$$

where we have defined the functions

$$(13) \quad b[u_a, u_b] = \frac{[u_a - u_b]}{[u_a - u_b + \eta]}, \quad c[u_a, u_b] = \frac{[\eta]}{[u_a - u_b + \eta]}, \quad [u] \equiv \sinh(u)$$

The  $R$ -matrix satisfies unitarity, crossing symmetry and the crucial Yang-Baxter equation that is required for integrability, given by

$$(14) \quad R_{ab}(u_a, u_b)R_{ac}(u_a, u_c)R_{bc}(u_b, u_c) = R_{bc}(u_b, u_c)R_{ac}(u_a, u_c)R_{ab}(u_a, u_b)$$

which holds in  $\text{End}(h_a \otimes h_b \otimes h_c)$  for all  $u_a, u_b, u_c$ .

As we will see in Section 3, the elements of the  $R$ -matrix (12) are the weights of the vertices of the trigonometric six-vertex model. This is the origin of the connection of the two models. One can graphically represent the elements of (12) to obtain the six vertices of the trigonometric six-vertex model in Figure 2.

**2.7. The  $L$ -matrix.** The  $L$ -matrix of the XXZ spin chain is a local operator that depends on a single rapidity  $u_a$ , and acts in the auxiliary space  $h_a$ . Its entries are operators acting at the  $m$ -th lattice site, and identically everywhere else. It has the form

$$(15) \quad L_{am}(u_a) = \begin{pmatrix} [u_a + \frac{\eta}{2}\sigma_m^z] & [\eta]\sigma_m^- \\ [\eta]\sigma_m^+ & [u_a - \frac{\eta}{2}\sigma_m^z] \end{pmatrix}_a$$

Using the definition of the  $R$ -matrix and the  $L$ -matrix, (12) and (15) respectively, the local intertwining equation is given by

$$(16) \quad R_{ab}(u_a, u_b)L_{am}(u_a)L_{bm}(u_b) = L_{bm}(u_b)L_{am}(u_a)R_{ab}(u_a, u_b)$$

The proof of (16) is immediate, if one uses the matrix representations of  $\sigma_m^z, \sigma_m^+, \sigma_m^-$  to write

$$(17) \quad L_{am}(u_a) = \begin{pmatrix} [u_a + \frac{\eta}{2}] & 0 & 0 & 0 \\ 0 & [u_a - \frac{\eta}{2}] & [\eta] & 0 \\ 0 & [\eta] & [u_a - \frac{\eta}{2}] & 0 \\ 0 & 0 & 0 & [u_a + \frac{\eta}{2}] \end{pmatrix}_{am} = [u_a + \eta/2]R_{am}(u_a, \eta/2)$$

This means that the  $L$ -matrix is equal to the  $R$ -matrix  $R_{am}(u_a, z_m)$  with  $z_m = \eta/2$ , up to an overall multiplicative factor. Cancelling these common factors from (16), it becomes

$$(18) \quad R_{ab}(u_a, u_b)R_{am}(u_a, \eta/2)R_{bm}(u_b, \eta/2) = R_{bm}(u_b, \eta/2)R_{am}(u_a, \eta/2)R_{ab}(u_a, u_b)$$

which is simply a corollary of the Yang-Baxter equation (14).

**2.8. The monodromy matrix  $M$ .** The monodromy or  $M$ -matrix is a global operator that acts on all sites in the spin chain. It is constructed as an ordered direct product of the  $L$ -matrices that act on single sites,

$$(19) \quad M_a(u_a) = L_{a1}(u_a) \dots L_{aL}(u_a)\Omega_a(\theta)$$

where  $\Omega_a(\theta)$  is a twist matrix given by

$$(20) \quad \Omega_a(\theta) = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}_a$$

The monodromy matrix is essential in the algebraic Bethe Ansatz approach to the diagonalization of the Hamiltonian  $H$ . It is convenient to define an inhomogeneous version, as an ordered direct product of  $R$ -matrices  $R_{am}(u_a, z_m)$ ,

$$(21) \quad M_a(u_a, \{z\}_L) = R_{a1}(u_a, z_1) \dots R_{aL}(u_a, z_L)\Omega_a(\theta)$$

The variables  $\{z_1, \dots, z_L\}$  are parameters corresponding with the sites of the spin chain and the homogeneous monodromy matrix, given by (19), is recovered by setting  $z_m = \eta/2$  for all  $1 \leq m \leq L$ . The inclusion of the variables  $\{z_1, \dots, z_L\}$  simplifies many later calculations, even though it is the homogeneous limit which actually interests us. We write the inhomogeneous monodromy matrix in  $(2 \times 2)$  block form, by defining

$$(22) \quad M_a(u_a, \{z\}_L) = \begin{pmatrix} e^{i\theta}A(u_a) & e^{-i\theta}B(u_a) \\ e^{i\theta}C(u_a) & e^{-i\theta}D(u_a) \end{pmatrix}_a$$

where the matrix entries are operators that act in  $\mathcal{H} = h_1 \otimes \cdots \otimes h_L$ . To simplify the notation, we have omitted the dependence of the elements of the  $M$ -matrix on the quantum rapidities  $\{z_1, \dots, z_L\}$ . This dependence is implied from now on.

The operator entries of the monodromy matrix satisfy a set of commutation relations, which are determined by the equation

$$(23) \quad R_{ab}(u_a, u_b) M_a(u_a, \{z\}_L) M_b(u_b, \{z\}_L) = M_b(u_b, \{z\}_L) M_a(u_a, \{z\}_L) R_{ab}(u_a, u_b)$$

which is a direct consequence of the Yang-Baxter equation (14) and the property

$$(24) \quad [R_{ab}(u_a, u_b), \Omega_a(\theta) \Omega_b(\theta)] = 0$$

of the twist matrix. Typical examples of these commutation relations, which are particularly important in the algebraic Bethe Ansatz, are

$$(25) \quad B(u)B(v) = B(v)B(u)$$

$$(26) \quad [u - v + \eta]B(u)A(v) = [\eta]B(v)A(u) + [u - v]A(v)B(u)$$

$$(27) \quad [\eta]B(u)D(v) + [u - v]D(u)B(v) = [u - v + \eta]B(v)D(u)$$

In Section 3, we identify the operator entries of the monodromy matrix (22) with rows of vertices from the six-vertex model, see Figure 3.

**2.9. The transfer matrix  $T$ .** The transfer matrix  $T \left( u_a, \{z\}_L \right)$  is defined as the trace of the inhomogeneous monodromy matrix

$$(28) \quad T \left( u_a, \{z\}_L \right) = \text{Tr}_a M_a(u_a, \{z\}_L) = e^{i\theta} A(u_a) + e^{-i\theta} D(u_a)$$

The Hamiltonian (10) is recovered via the quantum trace identity

$$(29) \quad H = [\eta] \frac{d}{du} \log T(u) \Big|_{u=\frac{\eta}{2}}, \quad \text{where} \quad T(u) = T \left( u, \{z\}_L \right) \Big|_{z_1=\dots=z_L=\frac{\eta}{2}}$$

where the anisotropy parameter in (10) is defined as  $\Delta = \cosh(\eta)$ . In this equation all quantum parameters have been set equal, so for the purpose of reconstructing the Hamiltonian  $H$  we see that the homogeneous monodromy matrix is sufficient. However, in all subsequent calculations we preserve the variables  $\{z_1, \dots, z_L\}$  and seek eigenvectors of  $T \left( u, \{z\}_L \right)$ . By (29), they are clearly also eigenvectors of  $H$ .

**2.10. Generic states, eigenstates and Bethe equations.** The initial and final spin-up reference states  $|L^\wedge\rangle$  and  $\langle L^\wedge|$  are eigenstates of the diagonal elements of the monodromy matrix. They satisfy the equations

$$(30) \quad A(u, \{z\}_L) |L^\wedge\rangle = a(u) |L^\wedge\rangle, \quad D(u, \{z\}_L) |L^\wedge\rangle = d(u) |L^\wedge\rangle$$

$$(31) \quad \langle L^\wedge| A(u, \{z\}_L) = a(u) \langle L^\wedge|, \quad \langle L^\wedge| D(u, \{z\}_L) = d(u) \langle L^\wedge|$$

where we have defined the eigenvalues

$$(32) \quad a(u) = 1, \quad d(u) = \prod_{i=1}^L \frac{[u - z_i]}{[u - z_i + \eta]}$$

This makes  $|L^\wedge\rangle$  and  $\langle L^\wedge|$  eigenstates of the transfer matrix  $T \left( u, \{z\}_L \right)$ . The rest of the eigenstates  $\{|\mathcal{O}\rangle\}$  of  $T \left( u, \{z\}_L \right)$ , that is, states that satisfy

$$(33) \quad T \left( u, \{z\}_L \right) |\mathcal{O}\rangle_\beta = \left( e^{i\theta} A(u) + e^{-i\theta} D(u) \right) |\mathcal{O}\rangle_\beta = E_{\mathcal{O}}(u) |\mathcal{O}\rangle_\beta$$

where  $E_{\mathcal{O}}(u)$  is the corresponding eigenvalue, are generated using the Bethe Ansatz. This is the statement that all eigenstates of  $T \left( u, \{z\}_L \right)$  are created in two steps. **1.** One acts on the initial reference state with the  $B$ -element of the monodromy matrix

$$(34) \quad |\mathcal{O}\rangle_\beta = B(u_{\beta_N}) \cdots B(u_{\beta_1}) |L^\wedge\rangle$$



where  $N \leq L$ , since acting on  $|L^\wedge\rangle$  with more  $B$ -operators than the number of sites in the spin chain annihilates it. This generates a ‘generic Bethe state’. **2.** We require that the auxiliary space rapidity variables  $\{u_{\beta_1}, \dots, u_{\beta_N}\}$  satisfy Bethe equations, hence the use of the subscript  $\beta$ <sup>13</sup>. We call the resulting state a ‘Bethe eigenstate’. That is,  $|\mathcal{O}\rangle_\beta$  is an eigenstate of  $T \left( u, \{z\}_L \right)$  if and only if

$$(35) \quad \frac{a(u_{\beta_i})}{d(u_{\beta_i})} = \prod_{j=1}^L \frac{[u_{\beta_i} - z_j + \eta]}{[u_{\beta_i} - z_j]} = e^{-2i\theta} \prod_{j \neq i}^N \frac{[u_{\beta_j} - u_{\beta_i} - \eta]}{[u_{\beta_j} - u_{\beta_i} + \eta]},$$

for all  $1 \leq i \leq N$ . This fact can be proved using the commutation relations (26) and (27), as well as (30) and (31). As remarked earlier, by virtue of (29), eigenstates of the transfer matrix  $T \left( u, \{z\}_L \right)$  are also eigenstates of the spin-chain Hamiltonian  $H$ . The latter is the spin-chain version of the 1-loop dilatation operator in SYM<sub>4</sub>. We construct eigenstates of  $T \left( u, \{z\}_L \right)$  in  $\mathcal{H}^*$  using the  $C$ -element of the  $M$ -matrix

$$(36) \quad {}_\beta \langle \mathcal{O} | = \langle L^\wedge | C(u_{\beta_1}) \dots C(u_{\beta_N})$$

where  $N \leq L$  to obtain a non vanishing result, and requiring that the auxiliary space rapidity variables satisfy the Bethe equations.

**2.11. Scalar products that are determinants.** Following [32, 33] we define the scalar product  $S[L, N_1, N_2]$ ,  $0 \leq N_2 \leq N_1$ , that involves  $(N_1 + N_2)$  operators,  $N_1$   $B$ -operators with auxiliary rapidities that satisfy Bethe equations, and  $N_2$   $C$ -operators with auxiliary rapidities that are free<sup>14</sup>. For  $N_2 = 0$ , we obtain, up to a non-dynamical factor, the domain wall partition function. For  $N_2 = N_1$ , we obtain Slavnov’s scalar product [35]. As we will see in Section 3,  $S[L, N_1, N_2]$  is the partition function (weighted sum over all internal configurations) of a lattice in an  $[L, N_1, N_2]$ -configuration, see Figure 9.

Let  $\{u_\beta\}_{N_1} = \{u_{\beta_1}, \dots, u_{\beta_{N_1}}\}$ ,  $\{v\}_{N_2} = \{v_1, \dots, v_{N_2}\}$ ,  $\{z\}_L = \{z_1, \dots, z_L\}$  be three sets of variables the first of which satisfies Bethe equations,  $0 \leq N_2 \leq N_1$  and  $1 \leq N_1 \leq L$ . We define the scalar products

$$(37) \quad S[L, N_1, N_2] \left( \{u_\beta\}_{N_1}, \{v\}_{N_2}, \{z\}_L \right) = \langle N_3^\vee, (L - N_3)^\wedge | \prod_{i=1}^{N_2} \mathbb{C}(v_i) \prod_{j=1}^{N_1} \mathbb{B}(u_{\beta_j}) | L^\wedge \rangle$$

with  $N_3 = N_1 - N_2$ , and where we have defined the normalized  $B$ - and  $C$ -operators

$$(38) \quad \mathbb{B}(u) = \frac{B(u)}{d(u)}, \quad \mathbb{C}(v) = \frac{C(v)}{d(v)}$$

which are introduced only as a matter of convention. It is clear that for  $N_2 = 0$ , we obtain a domain wall partition function, while for  $N_2 = N_1$ , we obtain Slavnov’s scalar product. In all cases, we assume that the auxiliary rapidities  $\{u_\beta\}_{N_1}$  obey the Bethe equations (35), and use the subscript  $\beta$  to emphasize that, while the auxiliary rapidities  $\{v\}_{N_2}$  are either free or also satisfy their own set of Bethe equations. When the latter is the case, this fact is not used. The quantum rapidities  $\{z\}_L$  are taken to be equal to the same constant value in the homogeneous limit.

<sup>13</sup>We use  $\beta$  in two different ways. **1.** To indicate the deformation parameter in SYM<sub>4</sub> <sup>$\beta$</sup>  theories, and **2.** To indicate that a certain state is a Bethe eigenstate of the spin-chain Hamiltonian. There should be no confusion with **1**, in which  $\beta$  is a parameter but never a subscript, while in **2** it is always a subscript.

<sup>14</sup>To simplify the notation, we use  $N_1$ ,  $N_2$  and  $N_3 = N_1 - N_2$ , instead of the corresponding notation used in [32, 33]. These variables match the corresponding ones in Section 4.

**2.12. A determinant expression for the Slavnov scalar product  $S[L, N_1, N_2]$ .** Following [32, 33], we consider the  $(N_1 \times N_1)$  matrix

$$(39) \quad \mathcal{S} \left( \{u_\beta\}_{N_1}, \{v\}_{N_2}, \{z\}_L \right) = \begin{pmatrix} f_1(z_1) & \cdots & f_1(z_{N_3}) & g_1(v_{N_2}) & \cdots & g_1(v_1) \\ \vdots & & \vdots & \vdots & & \vdots \\ f_{N_1}(z_1) & \cdots & f_{N_1}(z_{N_3}) & g_{N_1}(v_{N_2}) & \cdots & g_{N_1}(v_1) \end{pmatrix}$$

whose entries are the functions

$$(40) \quad f_i(z_j) = \left( \frac{[\eta]}{[u_{\beta_i} - z_j + \eta][u_{\beta_i} - z_j]} \right) \prod_{k=1}^{N_2} \frac{1}{[v_k - z_j]}$$

$$(41) \quad g_i(v_j) = \left( \frac{[\eta]}{[u_{\beta_i} - v_j]} \right) \times \left( \left( \prod_{k=1}^L \frac{[v_j - z_k + \eta]}{[v_j - z_k]} \prod_{k \neq i}^{N_1} [u_{\beta_k} - v_j + \eta] \right) - e^{-2i\theta} \prod_{k \neq i}^{N_1} [u_{\beta_k} - v_j - \eta] \right)$$

and where  $N_3 = N_1 - N_2$ . Since the auxiliary rapidities  $\{u_\beta\}_{N_1}$  satisfy Bethe equations (35), following [32, 33] it is possible to show that

$$(42) \quad S[L, N_1, N_2] = \frac{\prod_{i=1}^{N_1} \prod_{j=1}^{N_3} [u_{\beta_i} - z_j + \eta] \det \mathcal{S} \left( \{u_\beta\}_{N_1}, \{v\}_{N_2}, \{z\}_L \right)}{\prod_{1 \leq i < j \leq N_1} [u_{\beta_j} - u_{\beta_i}] \prod_{1 \leq i < j \leq N_2} [v_i - v_j] \prod_{1 \leq i < j \leq N_3} [z_i - z_j]}$$

**2.13. The Slavnov scalar product  $S[L, N_1, N_1]$ .** Consider the special case  $N_1 = N_2 = N$ , which corresponds to Slavnov's scalar product itself. In this case we obtain the  $(N \times N)$  matrix

$$(43) \quad \mathcal{S} \left( \{u_\beta\}_N, \{v\}_N, \{z\}_L \right) = \begin{pmatrix} g_1(v_N) & \cdots & g_1(v_1) \\ \vdots & & \vdots \\ g_N(v_N) & \cdots & g_N(v_1) \end{pmatrix}$$

whose entries are the functions

$$(44) \quad g_i(v_j) = \left( \frac{[\eta]}{[u_{\beta_i} - v_j]} \right) \times \left( \left( \prod_{k=1}^L \frac{[v_j - z_k + \eta]}{[v_j - z_k]} \prod_{k \neq i}^N [u_{\beta_k} - v_j + \eta] \right) - e^{-2i\theta} \prod_{k \neq i}^N [u_{\beta_k} - v_j - \eta] \right)$$

The Slavnov scalar product  $S[L, N, N]$  is then given by

$$(45) \quad S[L, N, N] = \frac{\det \mathcal{S} \left( \{u_\beta\}_N, \{v\}_N, \{z\}_L \right)}{\prod_{1 \leq i < j \leq N} [u_{\beta_i} - u_{\beta_j}] \prod_{1 \leq i < j \leq N} [v_i - v_j]}$$

**2.14. Restrictions.** There is a simple relation between the scalar products  $S[L, N_1, N_1]$  and  $S[L, N_1, N_2]$ , which was used in [33] to provide a recursive proof of Slavnov's scalar product formula [35]. It is easy to show that by restricting the free variables  $v_{N_1}, \dots, v_{N_2+1}$  in (45) to the values  $z_1, \dots, z_{N_3}$ , one obtains the recursion relation

$$(46) \quad \left( \prod_{i=N_2+1}^{N_1} \prod_{j=1}^L [v_i - z_j] S[L, N_1, N_1] \right) \bigg|_{\substack{v_{N_1}=z_1 \\ \vdots \\ v_{(N_2+1)}=z_{N_3}}} = \prod_{i=1}^{N_3} \prod_{j=1}^L [z_i - z_j + \eta] S[L, N_1, N_2]$$

As we show in Section 3, the scalar products  $S[L, N_1, N_1]$  and  $S[L, N_1, N_2]$  are in direct correspondence with the partition function of an  $[L, N_1, N_1]$ - and  $[L, N_1, N_2]$ -configuration, respectively. Accordingly, we expect that the recursion relation (46) has a natural interpretation at the level of six-vertex model lattice configurations, and indeed this turns out to be the case.

**2.15. The homogeneous limit of  $S[L, N_1, N_2]$ .** For the result in this paper, we need the homogeneous limit of  $S[L, N_1, N_2]$ , which we denote by  $S^{hom}[L, N_1, N_2]$ . Taking the limit  $z_i \rightarrow z$ ,  $i \in \{1, \dots, L\}$ , the result is

$$(47) \quad S^{hom}[L, N_1, N_2] = \frac{\prod_{i=1}^{N_1} [u_{\beta_i} - z + \eta]^{N_3} \det \mathcal{S}^{hom} \left( \{u_\beta\}_{N_1}, \{v\}_{N_2}, z \right)}{\prod_{1 \leq i < j \leq N_1} [u_{\beta_j} - u_{\beta_i}] \prod_{1 \leq i < j \leq N_2} [v_i - v_j]}$$

$$(48) \quad \mathcal{S}^{hom} \left( \{u_\beta\}_{N_1}, \{v\}_{N_2}, z \right) = \begin{pmatrix} \Phi_1^{(0)}(z) & \cdots & \Phi_1^{(N_3-1)}(z) & g_1^{hom}(v_{N_2}) & \cdots & g_1^{hom}(v_1) \\ \vdots & & \vdots & \vdots & & \vdots \\ \Phi_{N_1}^{(0)}(z) & \cdots & \Phi_{N_1}^{(N_3-1)}(z) & g_{N_1}^{hom}(v_{N_2}) & \cdots & g_{N_1}^{hom}(v_1) \end{pmatrix}$$

where  $\Phi_i^{(j)} = \frac{1}{j!} \partial_z^{(j)} f_i(z)$ , and

$$(49) \quad g_i^{hom}(v_j) = \frac{[\eta]}{[u_{\beta_i} - v_j]} \left[ \left( \frac{[v_j - z + \eta]}{[v_j - z]} \right)^L \prod_{k \neq i}^{N_1} [u_{\beta_k} - v_j + \eta] - e^{-2i\theta} \prod_{k \neq i}^{N_1} [u_{\beta_k} - v_j - \eta] \right]$$

**2.16. The Gaudin norm.** Let us consider the original, unrestricted Slavnov scalar product in the homogeneous limit  $z_i \rightarrow z$ ,  $S[L, N_1, N_1] \left( \{u_\beta\}_{N_1}, \{v\}_{N_1}, z \right)$ , and set  $\{v\}_{N_1} = \{u_\beta\}_{N_1}$  to obtain the Gaudin norm  $\mathcal{N} \left( \{u_\beta\}_{N_1} \right)$  which is the square of the norm of the Bethe eigenstate with auxiliary rapidities  $\{u_\beta\}_{N_1}$ . It inherits a determinant expression that can be computed starting from that of the Slavnov scalar product that we begin with and taking the limit  $\{v\}_{N_1} \rightarrow \{u_\beta\}_{N_1}$ . Following [32], one obtains

$$(50) \quad \mathcal{N} \left( \{u_\beta\}_{N_1} \right) = \left( e^{-2i\theta} [\eta] \right)^{N_1} \left( \prod_{i \neq j}^{N_1} \frac{[u_i - u_j + \eta]}{[u_i - u_j]} \right) \det \Phi' \left( \{u_\beta\}_{N_1} \right)$$

where

$$(51) \quad \Phi'_{ij} \left( \{u_\beta\}_{N_1} \right) = -\partial_{u_j} \ln \left[ \left( \frac{[u_i - z + \eta]}{[u_i - z]} \right)^L \prod_{\substack{k=1 \\ k \neq i}}^{N_1} \frac{[u_k - u_i + \eta]}{[u_k - u_i - \eta]} \right]$$

We need the Gaudin norm to normalize the Bethe eigenstates that form the 3-point functions whose structure constants we are interested in.

### 3. THE TRIGONOMETRIC SIX-VERTEX MODEL

This section follows almost *verbatim* the exposition in [21], up to straightforward adjustments to account for the fact that here we are interested in the trigonometric, rather than the rational six-vertex model. We recall the 2-dimensional trigonometric six-vertex model in the absence of external fields. From now on, ‘six-vertex model’ refers to that. It is equivalent to the XXZ spin- $\frac{1}{2}$  chain that appears in [20], but affords a diagrammatic representation that suits our purposes. We introduce quite a few terms to make this correspondence clear and the presentation precise, but the reader with basic familiarity with exactly solvable lattice models can skip all these.

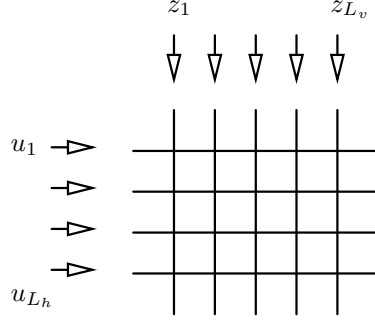


Figure 1: A square lattice with oriented lines and rapidity variables. Lattice lines are assigned the orientations indicated by the white arrows.

**3.1. Lattice lines, orientations, and rapidity variables.** Consider a square lattice with  $L_h$  horizontal lines and  $L_v$  vertical lines that intersect at  $L_h \times L_v$  points. There is no restriction, at this stage, on  $L_h$  or  $L_v$ . We order the horizontal lines from top to bottom and assign the  $i$ -th line an orientation from left to right and a rapidity variable  $u_i$ . We order the vertical lines from left to right and assign the  $j$ -th line an orientation from top to bottom and a rapidity variable  $z_j$ . See Figure 1. The orientations that we assign to the lattice lines are matters of convention and are only meant to make the vertices of the six-vertex model, that we introduce shortly, unambiguous. We orient the vertical lines from top to bottom to agree with the direction of the ‘spin set evolution’ that we introduce shortly.

**3.2. Line segments, arrows, and vertices.** Each lattice line is split into segments by all other lines that are perpendicular to it. ‘Bulk segments’ are attached to two intersection points, and ‘boundary segments’ are attached to one intersection point only. Assign each segment an arrow that can point in either direction, and define the vertex  $v_{ij}$  as the union of **1**. The intersection point of the  $i$ -th horizontal line and the  $j$ -th vertical line, **2**. The four line segments attached to this intersection point, and **3**. The arrows on these segments (regardless of their orientations). Assign  $v_{ij}$  a weight that depends on the specific orientations of its arrows, and the rapidities  $u_i$  and  $z_j$  that flow through it.

**3.3. Six vertices that conserve ‘arrow flow’.** Since every arrow can point in either direction, there are  $2^4 = 16$  possible types of vertices. In this note, we are interested in a model such that only those vertices that conserves ‘arrow flow’ (that is, the number of arrows that point toward the intersection point is equal to the number of arrows that point away from it) have non-zero weights. There are six such vertices. They are shown in Figure 2. We assign these vertices non-vanishing weights. We assign the rest of the 16 possible vertices zero weights [41].

In the trigonometric six-vertex model, and in the absence of external fields, the six vertices with non-zero weights form three equal-weight pairs of vertices, as in Figure 2. Two vertices that form a pair are related by reversing all arrows, thus the vertex weights are invariant under reversing all arrows. In the notation of Figure 2, the weights of the trigonometric six-vertex model, in the absence of external fields, are

$$(52) \quad a[u_i, z_j] = 1, \quad b[u_i, z_j] = \frac{[u_i - z_j]}{[u_i - z_j + \eta]}, \quad c[u_i, z_j] = \frac{[\eta]}{[u_i - z_j + \eta]}$$

where we use the definition  $[x] = \sinh(x)$  to simplify notation<sup>15</sup>. The assignment of weights in (52) satisfies unitarity, crossing symmetry, and most importantly the Yang-Baxter equations

<sup>15</sup>The weights of the six-vertex model (52) and the entries of the XXZ  $R$ -matrix (12) are identical. This is the origin of the connection between the two models. We have chosen to write down these functions twice for clarity and to emphasize this fact.

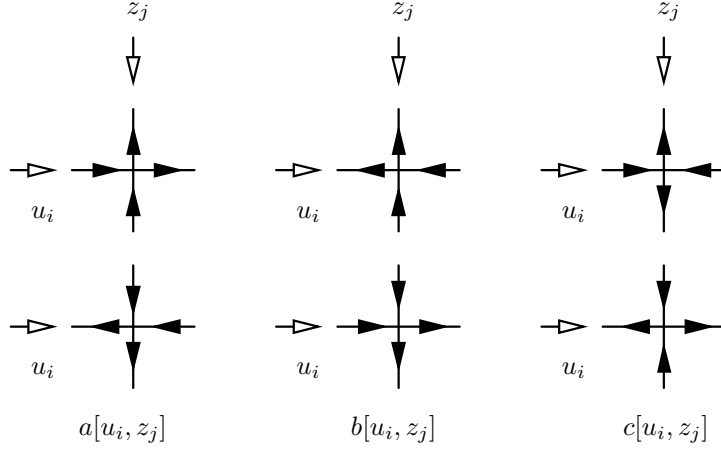


Figure 2: The non-vanishing-weight vertices of the six-vertex model. Pairs of vertices in the same column share the weight that is shown below that column. The white arrows indicate the line orientations needed to specify the vertices without ambiguity.

[41]. It is not unique since one can multiply all weights by the same factor without changing the final physical results.

**3.4. Correspondence with the XXZ  $R$ -matrix.** The connection with the  $R$ -matrix of the XXZ spin- $\frac{1}{2}$  chain is straightforward. One can think of the  $R$ -matrix (12) as assigning a weight to the transition from a pair of initial spin states (for example, the definite spin states on the right and upper segments that meet at a certain vertex) to a pair of final spin states (the definite spin states on the left and lower segments that meet at the same vertex as the initial ones). In the case of the trigonometric XXZ spin- $\frac{1}{2}$  chain, this is a transition between four possible initial spin states and four final spin states, and accordingly the  $R$ -matrix is  $(4 \times 4)$ . The six non-zero entries of (12) correspond with the vertices in Figure 2.

**3.5. Remarks. 1.** The spin chains that are relevant to integrability in YM theories are typically homogeneous since all quantum rapidities are set equal to the same constant value  $z$ . In our conventions,  $z = \frac{1}{2}\sqrt{-1}$ . **2.** The trigonometric six-vertex model that corresponds to the homogeneous XXZ spin- $\frac{1}{2}$  chain used in [20] has, in our conventions, all vertical rapidity variables equal to  $\frac{1}{2}\sqrt{-1}$ . In this note, we start with inhomogeneous vertical rapidities, then take the homogeneous limit at the end. **3.** In a 2-dimensional vertex model with no external fields, the horizontal lines are on equal footing with the vertical lines. To make contact with spin chains, we treat these two sets of lines differently. **4.** In all figures in this note, a line segment with an arrow on it obviously indicates a definite arrow assignment. A line segment with no arrow on it implies a sum over both arrow assignments.

**3.6. Weighted configurations and partition functions.** By assigning every vertex  $v_{ij}$  a weight  $w_{ij}$ , a vertex model lattice configuration with a definite assignment of arrows is assigned a weight equal to the product of the weights of its vertices. The partition function of a lattice configuration is the sum of the weights of all possible configurations that the vertices can take and that respect the boundary conditions. Since the vertex weights are invariant under reversal of all arrows, the partition function is also invariant under reversal of all arrows.

**3.7. Rows of segments, spin systems, spin system states and net spin.** A ‘row of segments’ is a set of *vertical* line segments that start and/or end on the same horizontal line(s). An  $L_h \times L_v$  six-vertex lattice configuration has  $(L_v + 1)$  rows of segments. On every length- $L_h$  row of segments, one can assign a definite spin configuration, whereby each segment carries a spin variable (an arrow) that can point either up or down. A ‘spin system’ on a specific row of

segments is a set of all possible definite spin configurations that one can assign to that row. A ‘spin system state’ is one such definite configuration. Two neighbouring spin systems (or spin system states) are separated by a horizontal lattice line. The spin systems that live on the top and the bottom rows of segments are initial and final spin systems, respectively. Consider a specific spin system state. Assign each up-spin the value  $+1$  and each down-spin the value  $-1$ . The sum of these values is the net spin of this spin system state. In this paper, we only consider six-vertex model configurations such that all elements in a spin system have the same net spin.

**3.8. Initial and final spin-up and spin-down reference states, and a variation.** An initial (final) spin-up reference state  $|L^\wedge\rangle$  ( $\langle L^\wedge|$ ) is a spin system state on a top (bottom) row of segments with  $L$  arrows that are all up. An initial (final) spin-down reference state  $|L^\vee\rangle$  ( $\langle L^\vee|$ ) is a spin system state on a top (bottom) row of segments with  $L$  arrows that are all down. The state  $\langle N_3^\vee, (L - N_3)^\wedge|$  is a spin system state on a bottom row of segments with  $L$  arrows such that the first  $N_3$  arrows from the left are down, while the remaining  $(L - N_3)$  arrows are up. We do not need the initial version of this state.

**3.9. Correspondence with XXZ spin chain states.** The connection to the XXZ spin- $\frac{1}{2}$  chain of Section 2 is clear. Every state of the periodic spin chain is analogous to a spin system state in the six-vertex model. Periodicity is not manifest in the latter representation for the same reason that it is not manifest once we choose a labeling system. The initial and final spin-up/down reference states are the six-vertex analogues of those discussed in Section 2.

**3.10. Remarks. 1.** There is of course no ‘time variable’ in the six-vertex model, but one can think of a spin system as a dynamical system that evolves in discrete steps as one scans a lattice configuration from top to bottom. Starting from an initial spin set and scanning the configuration from top to bottom, the intermediate spin sets are consecutive states in the history of a dynamical system, ending with the final spin set. This evolution is caused by the action of the horizontal line elements. **2.** In this paper, all elements in a spin system, that live on a certain row of segments, have the same net spin. The reason is that vertically adjacent spin systems are separated by horizontal lines of a fixed type that change the net spin by the same amount ( $\pm 1$ ) or keep it unchanged. Since we consider only lattice configurations with given horizontal lines (and do not sum over different types), the net spin of all elements in a spin system changes by the same amount.

**3.11. Four types of horizontal lines.** Each horizontal line has two boundary segments. Each boundary segment has as an arrow that can point into the configuration or away from it. Accordingly, we can distinguish four types of horizontal lines, as in Figure 3. We refer to them as  $A$ -,  $B$ -,  $C$ - and  $D$ -lines.

An important property of a horizontal line is how the net spin changes as one moves across it from top to bottom. Given that all vertices conserve ‘arrow flow’, one can easily show that, scanning a configuration from top to bottom,  $B$ -lines change the net spin by  $-1$ ,  $C$ -lines change it by  $+1$ , while  $A$ - and  $D$ -lines preserve the net spin. This can be easily understood by working out a few simple examples.

**3.12. Correspondence with monodromy matrix entries.** The  $A$ -,  $B$ -,  $C$ - and  $D$ -lines in Figure 3 are the six-vertex model representation of the corresponding elements of the  $M$ -matrix in Section 2. This graphical representation is used frequently throughout the rest of the paper.

**3.13. Four types of configurations. 1.** A  $B$ -configuration is a lattice configuration with  $L$  vertical lines and  $N$  horizontal lines,  $N \leq L$ , such that **A.** The initial spin system is an initial reference state  $|L^\wedge\rangle$ , and **B.** All horizontal lines are  $B$ -lines. An example is on the left hand side of Figure 4.

**2.** A  $C$ -configuration is a lattice configuration with  $L$  vertical lines and  $N$  horizontal lines,  $N \leq L$ , such that **A.** All horizontal lines are  $C$ -lines, and **B.** The final spin system is a final reference state  $\langle L^\wedge|$ . An example is on the right hand side of Figure 4.

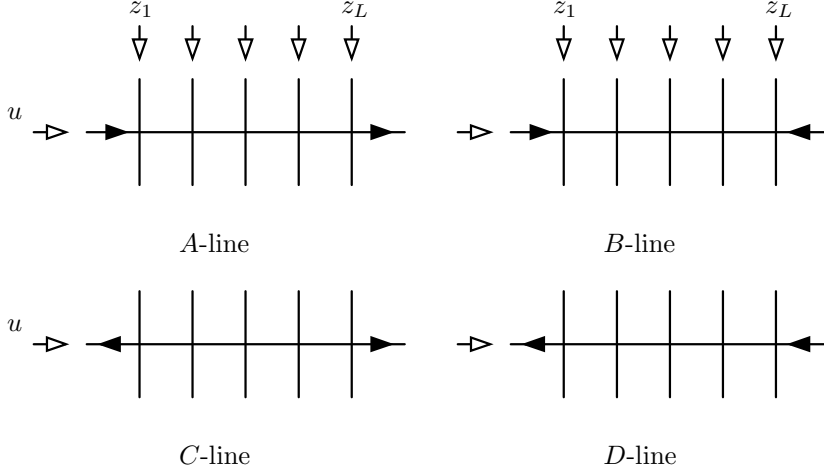


Figure 3: There are four types of horizontal lines in a six-vertex model lattice configuration.

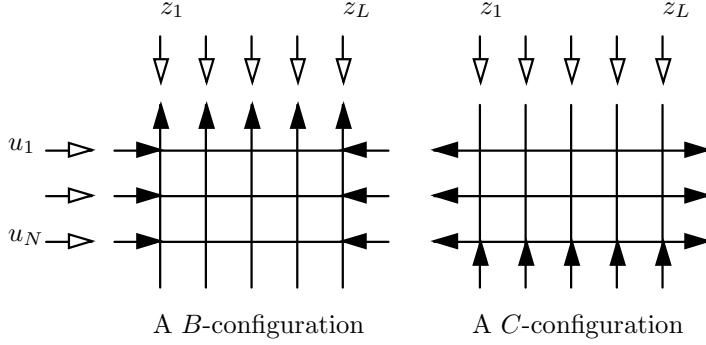


Figure 4: On the left, a  $B$ -configuration, generated by the action of  $N$   $B$ -lines on an initial length- $L$  reference state,  $N \leq L$ . A weighted sum over all possible configurations of segments with no arrows is implied. On the right, the corresponding  $C$ -configuration.

**3.** A  $BC$ -configuration is a lattice configuration with  $L$  vertical lines and  $2N_1$  horizontal lines,  $0 \leq N_1 \leq L$ , such that **A.** The initial spin system is an initial reference state  $|L^\wedge\rangle$ , **B.** The first  $N_1$  horizontal lines from top to bottom are  $B$ -lines, **C.** The following  $N_1$  horizontal lines are  $C$ -lines, **D.** The final spin system is a final reference state  $\langle L^\wedge|$ . See Figure 5<sup>16</sup>.

**4.** An  $[L, N_1, N_2]$ -configuration,  $0 \leq N_2 \leq N_1$ , is identical to a  $BC$ -configuration except that it has  $N_1$   $B$ -lines, and  $N_2$   $C$ -lines. When  $N_3 = N_1 - N_2 = 0$ , we evidently recover a  $BC$ -configuration. The case  $N_2 = 0$  is discussed below. For intermediate values of  $N_2$ , we obtain restricted  $BC$ -configurations whose partition functions turn out to be essentially the structure constants.

**3.14. Correspondence with generic Bethe states.** An initial (final) generic Bethe state is represented in six-vertex model terms as a  $B$ -configuration ( $C$ -configuration), as illustrated on the left (right) hand side of Figure 4. Note that the outcome of the action of the  $N$   $B$ -lines ( $C$ -lines) on the initial (final) length- $L$  spin-up reference state is an initial (final) spin system that can assume all possible spin states of net spin  $(L - N)$ . Each of these definite spin states is weighted by the weight of the corresponding lattice configuration.

<sup>16</sup>For visual clarity, we have allowed for a gap between the  $B$ -lines and the  $C$ -lines in Figure 5. There is also a gap between the  $N_3$ -th and  $(N_3 + 1)$ -th vertical lines, where  $N_3 = 3$  in the example shown, that indicates separate portions of the lattice that will be relevant shortly. The reader should ignore this at this stage.

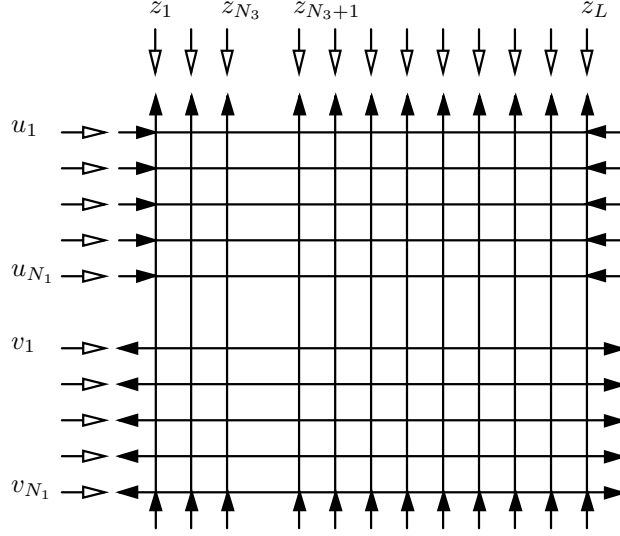


Figure 5: A six-vertex model  $BC$ -configuration.  $L = 12$ , and  $N_1 = 5$ , or equivalently  $L_h = 2 \times 5 = 10$  and  $L_v = 12$ . The top  $N_1$  horizontal lines represent  $B$ -operators. The bottom  $N_1$  horizontal lines represent  $C$ -operators. The initial (top) as well as the final (bottom) boundary spin systems are reference states.

**3.15. Correspondence with  $S[L, N_1, N_1]$  scalar products and  $S[L, N_1, N_2]$  restricted scalar products.** In the language of the six-vertex model, the scalar product  $S[L, N_1, N_1]$  corresponds with a  $BC$ -configuration with  $N_1$   $B$ -lines and  $N_1$   $C$ -lines, as illustrated in Figure 5. The restricted scalar product  $S[L, N_1, N_2]$  corresponds with an  $[L, N_1, N_2]$ -configuration, as illustrated in Figure 9. Compared with the definition of  $S[L, N_1, N_2]$  in (37), the partition function of an  $[L, N_1, N_2]$ -configuration differs only up to an overall normalization. To translate between the two, one should divide the latter by  $d(u)$  for every  $B$ -line with rapidity  $u$  and by  $d(v)$  for every  $C$ -line with rapidity  $v$ .

**3.16.  $[L, N_1, N_2]$ -configurations as restrictions of  $BC$ -configurations.** Consider a  $BC$ -configuration with no restrictions. To be specific, let us consider the configuration in Figure 5, where  $N_1 = 5$  and  $L = 12$ . Both sets of rapidities  $\{u\}$  and  $\{v\}$  are labeled from top to bottom, as usual.

Consider the vertex at the bottom-left corner of Figure 5. From Figure 2, it is easy to see that this can be either a  $b$ - or a  $c$ -vertex. Since the  $\{v\}$  variables are free, set  $v_5 = z_1$ , thereby setting the weight of all configurations with a  $b$ -vertex at this corner to zero, and forcing the vertex at this corner to be a  $c$ -vertex.

Referring to Figure 2 again, one can see that not only is the corner vertex forced to be a  $c$ -vertex, but the orientations of all arrows on the horizontal lattice line with rapidity  $v_5$ , as well as all arrows on the vertical line with rapidity  $z_1$  but below the horizontal line with rapidity  $u_{N_1}$  are also frozen to fixed values as in Figure 6.

The above exercise in ‘freezing’ vertices and arrows can be repeated and to produce a non-trivial example, we do it two more times. Setting  $v_4 = z_2$  forces the vertex at the intersection of the lines carrying the rapidities  $v_4$  and  $z_2$  to be a  $c$ -vertex and freezes all arrows to the right as well as all arrows above that vertex and along  $C$ -lines, as in Figure 7.

Setting  $v_3 = z_3$ , we end up with the lattice configuration in Figure 8, from which we can see that **1.** All arrows on the lower  $N_3$  horizontal lines, where  $N_3 = 3$  in the specific example shown, are frozen, and **2.** All lines on the  $N_3$  left most vertical lines in the lower half of the diagram, where they intersect with  $C$ -lines. Removing the lower  $N_3$   $C$ -lines we obtain the configuration in Figure 9. This configuration has a subset (rectangular shape on lower left



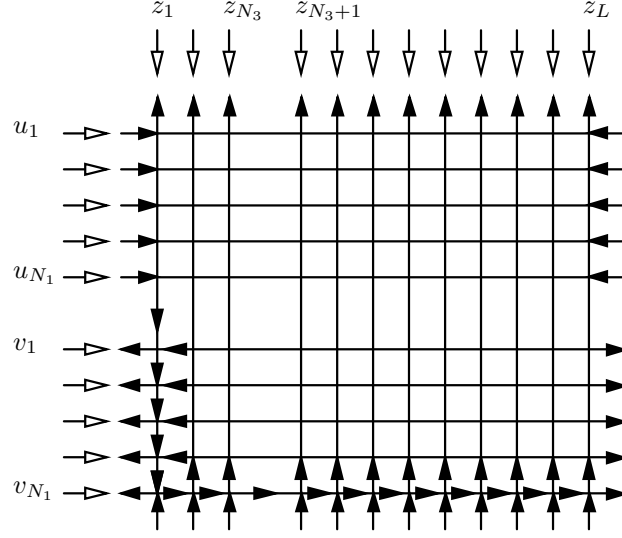


Figure 6: Setting  $v_{N_1}$  to  $z_1$  in Figure 5, we freeze **1.** the vertex at the lower left corner to be type- $c$ , **2.** all vertices to the right of the frozen corner to be type- $a$ , and **3.** all vertices above the frozen corner, but on the lower half of the diagram, to be type- $b$ .

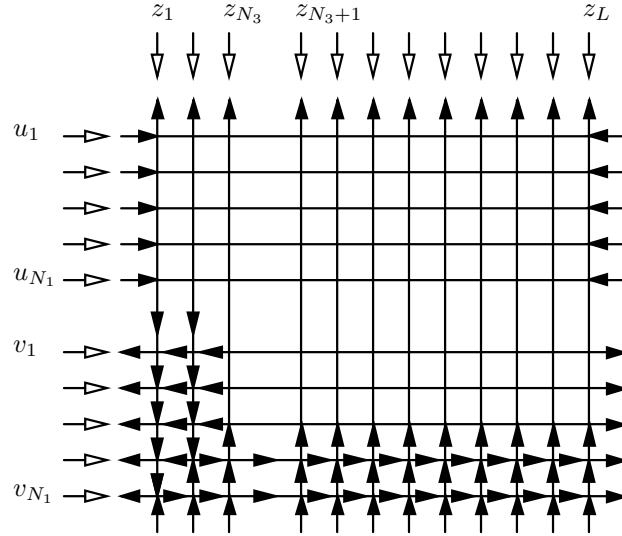


Figure 7: Setting  $v_{N_1-1}$  (on second horizontal line from below) to  $z_2$  (on second vertical line from left) in Figure 6, we freeze **1.** the vertex at the intersection of the lines that carry rapidities  $v_{N_1-1}$  and  $z_2$  to be type- $c$ , **2.** all vertices to the right of the most recently-frozen corner to be type- $a$ , and **3.** all vertices above the same vertex, but on the lower half of the diagram, to be type- $b$ .

corner) that is also completely frozen. All vertices in this part are  $a$ -vertices, hence from (52), their contribution to the partition function of this configuration is trivial.

An  $[L, N_1, N_2]$ -configuration, as in Figure 9, interpolates between an initial reference state  $|L^\wedge\rangle$  and a final  $\langle N_3^\vee, (L - N_3)^\wedge|$  state, using  $N_1$   $B$ -lines followed by  $N_2$   $C$ -lines.

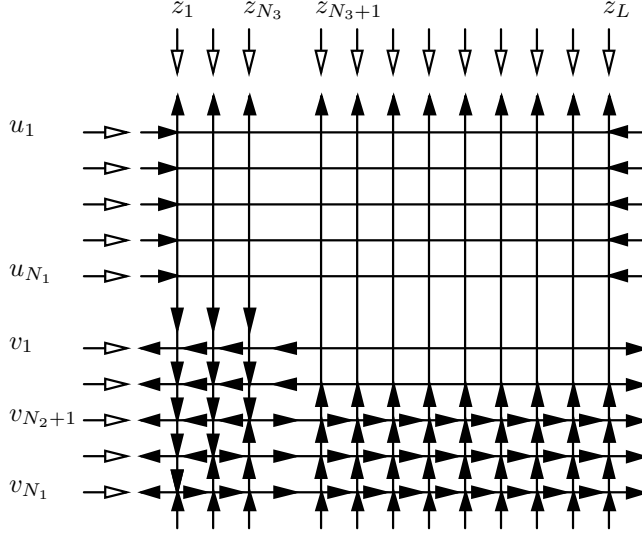


Figure 8: The effect of forcing the three vertices at the intersection of the lines that carry the pairs of rapidities  $\{v_{N_1}, z_1\}$ ,  $\{v_{N_1-1}, z_2\}$  and  $\{v_{N_1-2}, z_3\}$  to be  $c$ -vertices. We used the notation  $N_3 = N_1 - N_2$ .

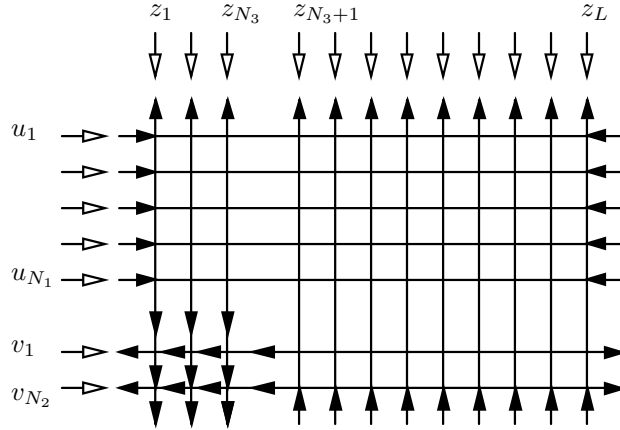


Figure 9: An  $[L, N_1, N_2]$ -configuration. In this example,  $N_1 = 5$ ,  $N_2 = 2$ , and as always  $N_3 = N_1 - N_2$ .

Setting  $v_{N_1-i+1} = z_i$  for  $i = 1, \dots, N_1$ , we freeze all arrows that are on  $C$ -lines or on segments that end on  $C$ -lines. Discarding these we obtain the lattice configuration in Figure 10.

Removing all frozen vertices (as well as the extra space between two sets of vertical lines, that is no longer necessary), one obtains the *domain wall configuration* in Figure 11, which is characterized as follows. All arrows on the left and right boundaries point inwards, and all arrows on the upper and lower boundaries point outwards. The internal arrows remain free, and the configurations that are consistent with the boundary conditions are summed over. Reversing the orientation of all arrows on all boundaries is a dual a domain wall configuration.

**3.17. Remarks on domain wall configurations. 1.** One can generate a domain wall configuration directly starting from a length- $N$  initial reference state followed by  $N$   $B$ -lines, **2.** One can generate a dual domain wall configuration directly starting from a length- $N$  dual initial

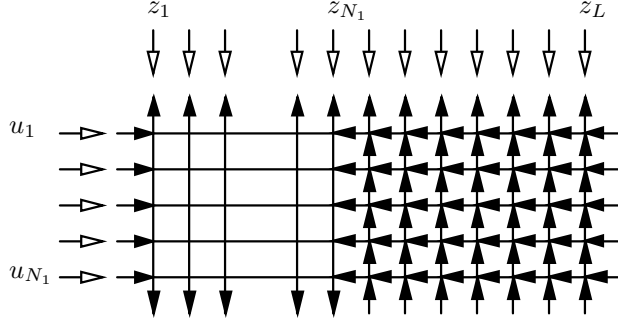


Figure 10: Another  $[L, N_1, N_2]$ -configuration. In this example,  $N_2 = 0$  and  $N_1 = 5$ . Equivalently, the left half is an  $(N_1 \times N_1)$  domain wall configuration, where  $N_1 = 5$ , with an additional totally frozen lattice configuration to its right.

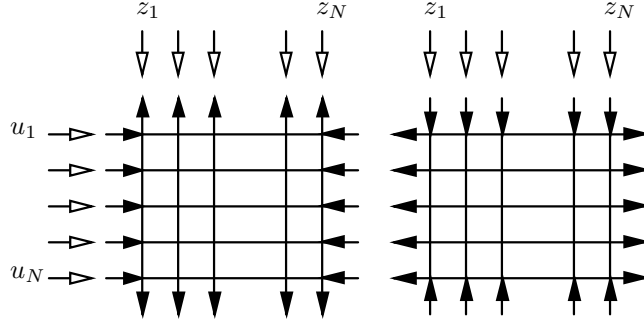


Figure 11: The left hand side is an  $(N \times N)$  domain wall configuration, where  $N = 5$ . The right hand side is the corresponding dual configuration.

reference state followed by  $N$   $C$ -lines, **3**. A  $BC$ -configuration with length- $L$  initial and final reference states,  $L$   $B$ -lines and  $L$   $C$ -lines, factorizes into a product of a domain wall configuration and a dual domain wall configuration, **4**. The restriction of  $BC$ -configurations to  $[L, N_1, N_2]$ -configurations, where  $N_2 < N_1$ , produces a recursion relation that was used in [33] to provide a recursive proof of Slavnov's determinant expression for the scalar product of a Bethe eigenstate and a generic state in the corresponding spin chain, **5**. The partition function of a domain wall configuration has a determinant expression found by Izergin, that can be derived in six-vertex model terms (without reference to spin chains or the BA) [40].

**3.18. Izergin's domain wall partition function.** Let  $\{u\}_N = \{u_1, \dots, u_N\}$  and  $\{z\}_N = \{z_1, \dots, z_N\}$  be two sets of rapidity variables<sup>17</sup> and define  $Z_N \left( \{u\}_N, \{z\}_N \right)$  to be the partition function of the domain wall lattice configuration on the left hand side of Figure 11, after dividing by  $d(u)$  for every  $B$ -line with rapidity  $u$ . Izergin's determinant expression for the domain wall partition function is

$$(53) \quad Z_N \left( \{u\}_N, \{z\}_N \right) = \frac{\prod_{i,j=1}^N [u_i - z_j + \eta]}{\prod_{1 \leq i < j \leq N} [u_i - u_j][z_j - z_i]} \det \left( \frac{[\eta]}{[u_i - z_j + \eta][u_i - z_j]} \right)_{1 \leq i,j \leq N}$$

<sup>17</sup>The following result does not require that any set of rapidities satisfy Bethe equations.

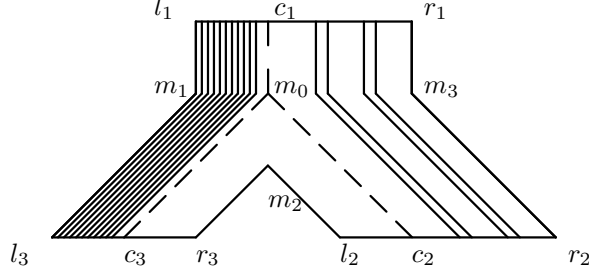


Figure 12: A schematic representation of a 3-point function. State  $\mathcal{O}_1$  is on top.  $\mathcal{O}_2$  and  $\mathcal{O}_3$  are below, to the right and to the left. Type-**A** 3-point functions are (initially) in this ‘wide-pants’ form.

Dual domain wall configurations have the same partition functions due to invariance under reversing all arrows. For the result of this note, we need the homogeneous limit of the above expression. Taking the limit  $z_i \rightarrow z$ ,  $\{i = 1, \dots, L\}$ , we obtain

$$(54) \quad Z_N^{hom}(\{u\}_N, z) = \frac{\prod_{i=1}^N [u_i - z + \eta]^N}{\prod_{1 \leq i < j \leq N} [u_i - u_j]} \det \left( \phi^{(j-1)}(u_i, z) \right)_{1 \leq i, j \leq N}$$

where

$$(55) \quad \phi^{(j)}(u_i, z) = \frac{1}{j!} \partial_z^{(j)} \left( \frac{[\eta]}{[u_i - z + \eta][u_i - z]} \right)$$

#### 4. STRUCTURE CONSTANTS IN TYPE-**A** THEORIES

In this section, we recall the discussion of  $\text{SYM}_4$  tree-level structure constants of [20, 21] but now in the context of the Type-**A** theories in Subsection **0.5**, and construct determinant expressions for structure constants of three non-extremal  $SU(2)$  single-trace operators.

Since theory **1** is  $\text{SYM}_4$ , theory **2** is an Abelian orbifolding of  $\text{SYM}_4$ , and theory **3** is a real- $\beta$ -deformation of it, all three theories share the same fundamental charged scalar field content, that is  $\{X, Y, Z\}$  and their charge conjugates  $\{\bar{X}, \bar{Y}, \bar{Z}\}$ , and all are conformally invariant up to all loops [25]. This makes our discussion a straightforward paraphrasing of that in [20, 21].

**4.1. Tree-level structure constants.** We consider tree-level 3-point functions of  $SU(2)$  single-trace operators that **1.** have well-defined conformal dimensions at 1-loop level, and **2.** can be mapped to Bethe eigenstates in closed spin- $\frac{1}{2}$  chains.

These 3-point functions can be represented schematically as in Figure **12**. Identify the pairs of corner points  $\{l_1, r_1\}$ ,  $\{l_2, r_2\}$ ,  $\{l_3, r_3\}$ , as well as the triple  $\{m_1, m_2, m_3\}$  to obtain a pants diagram. The structure constants have a perturbative expansion in the ‘t Hooft coupling constant  $\lambda$ ,

$$(56) \quad C_{ijk} = c_{ijk}^{(0)} + \lambda c_{ijk}^{(1)} + \dots$$

We restrict our attention to the leading coefficient  $c_{ijk}^{(0)}$ . In the limit  $\lambda \rightarrow 0$ , many single-trace operators have the same conformal dimension. This degeneracy is lifted at 1-loop level and certain linear combinations of single-trace operators have definite 1-loop conformal dimension. This is why although we compute tree-level structure constants, we insist on 1-loop conformal invariance: We identify operators with well defined conformal dimensions.

As explained in Section 1, these linear combinations correspond to eigenstates of a closed spin- $\frac{1}{2}$  chain. Their conformal dimensions are the corresponding Bethe eigenvalues. These closed spin chain states correspond to the circles at the boundaries of the pants diagram that can be constructed from Figure 12 as discussed above.

**4.2. Remark.** In computing 3-point functions, the three composite operators *may or may not belong to the same*  $SU(2)$  doublet. In particular, in [20], EGSV use operators from the doublets  $\{Z, X\}$ ,  $\{\bar{Z}, \bar{X}\}$ , and  $\{Z, \bar{X}\}$ . In [21], this procedure allowed us to construct determinant expressions for structure constants of non-extremal 3-point functions. This applies to all Type-**A** theories. Type-**B** structure constants are constructed differently. In particular, the non-extremal case  $l_{23} = 0$  is considered.

**4.3. Constructing 3-point functions.** To construct three-point functions at the gauge theory operator level, the fundamental fields in the operators  $\mathcal{O}_i$ ,  $i = \{1, 2, 3\}$  are contracted by free propagators. Each propagator connects two fields, hence  $L_1 + L_2 + L_3$  is an even number. The number of propagators between  $\mathcal{O}_i$  and  $\mathcal{O}_j$  is

$$(57) \quad l_{ij} = \frac{1}{2}(L_i + L_j - L_k)$$

where  $(i, j, k)$  take distinct values in  $(1, 2, 3)$ . We restrict our attention, in this section, to the non-extremal case, that is, all  $l_{ij}$ 's are strictly positive. The free propagators reproduce the factor  $1/|x_i - x_j|^{\Delta_i + \Delta_j - \Delta_k}$  in (2), where  $\Delta_i = \Delta_i^{(0)}$ , the tree-level conformal dimension. See Figure 12 for a schematic representation of a three point function of the type discussed in this note. The horizontal line segment between  $l_i$  and  $r_i$  represents the operator  $\mathcal{O}_i$ . The lines that start at  $\mathcal{O}_1$  and end at either  $\mathcal{O}_2$  or  $\mathcal{O}_3$  represent one type of propagator.

**4.4. From single-trace operators to spin-chain states.** One represents the single-trace operator  $\mathcal{O}_i$  of well-defined 1-loop conformal dimension  $\Delta_i$  by a closed spin-chain Bethe eigenstate  $|\mathcal{O}_i\rangle_\beta$ . Its eigenvalue  $E_i$  is equal to  $\Delta_i$ . The number of fundamental fields  $L_i$  in the trace is the length of the spin chain.

The single-trace operator  $\mathcal{O}_i$  is a composite operator built from weighted sums over traces of products of two fundamental fields  $\{u, d\}$ . These fundamental fields are mapped to definite spin states. To perform suitable mappings that lead to non-vanishing results, we need to decide on which state(s) are in-state(s) from the viewpoint of the lattice representation, and which are out-state(s).

**4.5. Type-A. Fundamental field content of the states.** All three Type-**A** theories have the same fundamental field content, namely that of  $\text{SYM}_4$ , and thereby, more than one doublet. We focus on the doublets formed from the fields  $Z, X$  and their conjugates. Following [20], we identify the fundamental field content of  $\mathcal{O}_i$ ,  $i \in \{1, 2, 3\}$  with spin-chain spin states as shown in Table 1, where  $\bar{Z}$  and  $\bar{X}$  are the conjugates of  $Z$  and  $X$ . That is, if  $Z$  appears on one side of a propagator and  $\bar{Z}$  appears on the other side, then that propagator is not identically vanishing, and  $Z$  and  $\bar{Z}$  can be Wick contracted. Similarly for  $X$  and  $\bar{X}$ .

Operator	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \end{pmatrix}$
$\mathcal{O}_1$	$Z$	$X$	$\bar{Z}$	$\bar{X}$
$\mathcal{O}_2$	$\bar{Z}$	$\bar{X}$	$Z$	$X$
$\mathcal{O}_3$	$Z$	$\bar{X}$	$\bar{Z}$	$X$

TABLE 1. Identification of operator content of  $\mathcal{O}_i$ ,  $i \in \{1, 2, 3\}$  with initial and final spin-chain states.

In our conventions

$$(58) \quad \langle \bar{Z}Z \rangle = \langle Z|Z \rangle = 1, \quad \langle ZZ \rangle = \langle \bar{Z}|Z \rangle = 0$$

and similarly for  $X$  and  $\bar{X}$ . In (58),  $\langle \bar{f} f \rangle$  with no vertical bar between the two operators is a propagator, while  $\langle f|f \rangle$  with a vertical bar between the two operators is a scalar product of an initial state  $|f\rangle$  and a final state  $\langle f|$ .

From Table 1, one can read the fundamental-scalar operator content of each single-trace operator  $\mathcal{O}_i$ ,  $i \in \{1, 2, 3\}$ , when it is an initial state and when it is a final state. For example, the fundamental field content of the initial state  $|\mathcal{O}_1\rangle$  is  $\{Z, X\}$ , and that of the corresponding final state  $\langle \mathcal{O}_1|$  is  $\{\bar{Z}, \bar{X}\}$ . The content of an initial state and the corresponding final state are related by the ‘flipping’ operation of [20] described below.

**4.6. Structure constants in terms of spin-chains.** Having mapped the single-trace operators  $\mathcal{O}_i$ ,  $i \in \{1, 2, 3\}$  to spin-chain eigenstates, EGSV construct the structure constants in three steps.

**Step 1. Split the lattice configurations that correspond to closed spin-chain eigenstates into two parts.** Consider the open 1-dimensional lattice configuration that corresponds to the  $i$ -th closed spin-chain eigenstate,  $i \in \{1, 2, 3\}$ . This is schematically represented by a line in Figure 12 that starts at  $l_i$  and ends at  $r_i$ . Split that, at point  $c_i$  into *left* and *right* sub-lattice configurations of lengths  $L_{i,l} = \frac{1}{2}(L_i + L_j - L_k)$  and  $L_{i,r} = \frac{1}{2}(L_i + L_k - L_j)$  respectively. Note that the lengths of the sub-lattices is fully determined by  $L_1$ ,  $L_2$  and  $L_3$  which are fixed.

Following [34], we express the single lattice configuration of the original closed spin chain state as a weighted sum of tensor products of states that live in two smaller Hilbert spaces. The latter correspond to closed spin chains of lengths  $L_{i,l}$  and  $L_{i,r}$  respectively. That is,  $|\mathcal{O}_i\rangle = \sum H_{l,r} |\mathcal{O}_i\rangle_l \otimes |\mathcal{O}_i\rangle_r$ . The factors  $H_{l,r}$  were computed in [34] and were needed in [20], where one of the scalar products is generic and had to be expressed as an explicit sum. They are not needed in this work as we use Bethe equations to evaluate this very sum as a determinant.

**Step 2. From initial to final states.** Map  $|\mathcal{O}_i\rangle_l \otimes |\mathcal{O}_i\rangle_r \rightarrow |\mathcal{O}_i\rangle_l \otimes_r \langle \mathcal{O}_i|$ , using the operator  $\mathcal{F}$  that acts as follows.

$$(59) \quad \mathcal{F} \left( |f_1 f_2 \cdots f_{L-1} f_L\rangle \right) = \langle \bar{f}_L \bar{f}_{L-1} \cdots \bar{f}_2 \bar{f}_1 |$$

In particular,

$$(60) \quad \langle ZZ \cdots Z | ZZ \cdots Z \rangle = \langle \bar{Z} \bar{Z} \cdots \bar{Z} | \bar{Z} \bar{Z} \cdots \bar{Z} \rangle = 1, \quad \text{and} \quad \langle \bar{Z} \bar{Z} \cdots \bar{Z} | ZZ \cdots Z \rangle = 0$$

More generally

$$(61) \quad \langle f_{i_1} f_{i_2} \cdots f_{i_L} | f_{j_1} f_{j_2} \cdots f_{j_L} \rangle \sim \delta_{i_1 j_1} \delta_{i_2 j_2} \cdots \delta_{i_L j_L}$$

The ‘flipping’ operation in (59) is the origin of the differences in assignments of fundamental fields to initial and final operator states in Table 1. For example,  $|\mathcal{O}_1\rangle$  has fundamental field content  $\{Z, X\}$ , but  $\langle \mathcal{O}_1|$  has fundamental field content  $\{\bar{Z}, \bar{X}\}$ . This agrees with the fact that in computing  $\langle \mathcal{O}_i | \mathcal{O}_i \rangle$ , free propagators can only connect conjugate fundamental fields.

**Step 3. Compute scalar products.** Wick contract pairs of initial states  $|\mathcal{O}_i\rangle_r$  and final states  $|\mathcal{O}_{i+1}\rangle_l$ , where  $i \in \{1, 2, 3\}$  and  $i+3 \equiv i$ . The spin-chain equivalent of that is to compute the scalar products  ${}_r \langle \mathcal{O}_i | \mathcal{O}_{i+1} \rangle_l$ , which in six-vertex model terms are *BC*-configurations. The most general scalar product that we can consider is the generic scalar product between two generic Bethe states

$$(62) \quad S_{generic}(\{u\}, \{v\}) = \langle 0 | \prod_{j=1}^N \mathbb{C}(v_j) \prod_{j=1}^N \mathbb{B}(u_j) | 0 \rangle$$

A computationally tractable evaluation of  $S_{generic}(\{u\}, \{v\})$  using the commutation relations of BA operators is known [36]. Simpler expressions are obtained when the auxiliary rapidities of one (or both) states satisfies Bethe equations. The result in this case is a determinant. When only one set satisfies Bethe equations, one obtains a Slavnov scalar product. This was discussed in Section 2.

**4.7. Type-A. An unevaluated expression.** The above three steps lead to the following preliminary, unevaluated expression

$$(63) \quad c_{123}^{(0)} = \mathcal{N}_{123} \sum r \langle \mathcal{O}_3 | \mathcal{O}_1 \rangle_l r \langle \mathcal{O}_1 | \mathcal{O}_2 \rangle_l r \langle \mathcal{O}_2 | \mathcal{O}_3 \rangle_l$$

where the normalization factor  $\mathcal{N}_{123}$ , that turns out to be a non-trivial object that depends on the norms of the Bethe eigenstates, is

$$(64) \quad \mathcal{N}_{123} = \sqrt{\frac{L_1 L_2 L_3}{\mathcal{N}_1 \mathcal{N}_2 \mathcal{N}_3}}$$

In (64),  $L_i$  is the number of sites in the closed spin chain that represents state  $\mathcal{O}_i$ .  $\mathcal{N}_i$  is the Gaudin norm of state  $\mathcal{O}_i$  as in (50). The sum in (63) is to be understood as follows. **1.** It is a sum over all possible ways to split the sites of each closed spin chain (represented as a segment in a 1-dimensional lattice) into a left part and a right part. We will see shortly that only one term in this sum survives. **2.** It is a sum over all possible ways of partitioning the  $X$  or  $\bar{X}$  content of a spin chain state between the two parts that that spin chain was split into. We will see shortly that only one sum survives.

**4.8. Type-A. Simplifying the unevaluated expression.** Wick contracting single-trace operators, we can only contract a fundamental field with its conjugate. Given the assignments in Table 1, one can see that **1.** All  $Z$  fields in  $\mathcal{O}_3$  must contract with  $\bar{Z}$  fields in  $\mathcal{O}_2$ . The reason is that there are  $\bar{Z}$  fields only in  $\mathcal{O}_2$ , and none in  $\mathcal{O}_1$ . **2.** All  $\bar{X}$  fields in  $\mathcal{O}_3$  contract with  $X$  fields in  $\mathcal{O}_1$ . The reason is that there are  $X$  fields only in  $\mathcal{O}_1$ , and none in  $\mathcal{O}_2$ . If the total number of scalar fields in  $\mathcal{O}_i$  is  $L_i$ , and the number of  $\{X, \bar{X}\}$ -type scalar fields is  $N_i$ , then

$$(65) \quad l_{13} = N_3, \quad l_{23} = L_3 - N_3, \quad l_{12} = L_1 - N_3$$

and we have the constraint

$$(66) \quad N_1 = N_2 + N_3$$

From (65) and (66), we have the following 4 simplifications. **1.** There is only one way to split each lattice configuration that represents a spin chain into a left part and a right part, **2.** The scalar product  $r \langle \mathcal{O}_2 | \mathcal{O}_3 \rangle_l$  involves the fundamental field  $Z$  (and only  $Z$ ) in the initial state  $|\mathcal{O}_3\rangle_l$  as well as in the final state  $r \langle \mathcal{O}_2|$ . Using Table 1, we find that these states translate to an initial and a final spin-up reference state, respectively. This is represented in Figure 12 by the fact that no connecting lines (that stand for propagators of  $\{X, \bar{X}\}$  states) connect  $\mathcal{O}_2$  and  $\mathcal{O}_3$ . The scalar product of the two reference states is  $r \langle \mathcal{O}_2 | \mathcal{O}_3 \rangle_l = 1$ , **3.** The scalar product  $r \langle \mathcal{O}_3 | \mathcal{O}_1 \rangle_l$  involves the fundamental fields  $X$  (and only  $X$ ) in the initial state  $|\mathcal{O}_1\rangle_l$  as well as in the final state  $r \langle \mathcal{O}_3|$ . Using Table 1, we find that these states translate to an initial spin-up and a final spin-down reference state, respectively. This is represented in Figure 12 by the high density of connecting lines (that stand for propagators of  $\{X, \bar{X}\}$  states) between  $\mathcal{O}_1$  and  $\mathcal{O}_3$ . This scalar product is straightforward to evaluate in terms of the domain wall partition function, **4.** In the remaining scalar product  $r \langle \mathcal{O}_1 | \mathcal{O}_2 \rangle_l$ , both the initial state  $|\mathcal{O}_2\rangle_l$

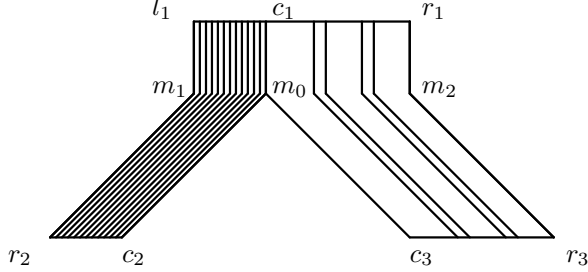


Figure 13: A schematic representation of a 3-point function after removal of a contraction between the left part of  $\mathcal{O}_2$  and the right part of  $\mathcal{O}_3$ , that evaluates to a factor of 1. Type-B 3-point functions are in this ‘narrow pants’ form from the outset.

and the final state  ${}_r\langle\mathcal{O}_1|$  involve  $\{\bar{X}, \bar{Z}\}$ . These fields translate to up and down spin states and the scalar product is generic. Using the BA commutation relations, it can be evaluated as a weighted sum [34].

**4.9. Type-A. Evaluating the expression.** The idea of [21] is to identify the expression in (63), up to simple factors, with the partition function of an  $[L_1, N_1, N_2]$ -configuration. Since this partition function is a restricted scalar product  $S[L_1, N_1, N_2]$ , it can be evaluated as a determinant. This is achieved in two steps.

**Step 1. Re-writing one of the scalar products.** We use the facts that **1.**  ${}_r\langle\mathcal{O}_2|\mathcal{O}_3\rangle_l = 1$ , and **2.**  ${}_r\langle\mathcal{O}_2|\mathcal{O}_1\rangle_l = {}_l\langle\mathcal{O}_1|\mathcal{O}_2\rangle_r$ , which is true for all scalar products, to re-write (63) in the form

$$(67) \quad c_{123}^{(0)} = \mathcal{N}_{123} \sum_{\alpha \cup \bar{\alpha} = \{u_\beta\}_{N_1}} {}_r\langle\mathcal{O}_3|\mathcal{O}_1\rangle_l {}_l\langle\mathcal{O}_2|\mathcal{O}_1\rangle_r = \mathcal{N}_{123} \left( {}_r\langle\mathcal{O}_3| \otimes {}_l\langle\mathcal{O}_2| \right) |\mathcal{O}_1\rangle$$

where the right hand side of (67) is a scalar product of the full initial state  $|\mathcal{O}_1\rangle$  (so we no longer have a sum over partitions of the rapidities  $\{u_\beta\}_{N_1}$  since we no longer split the state  $\mathcal{O}_1$ ) and two states that are pieces of original states that were split. Deleting the scalar product corresponding to contracting the left part of state  $\mathcal{O}_2$  with the right part of state  $\mathcal{O}_3$ , since that contraction leads to a factor of unity, the object that we are evaluating can be schematically drawn as in Figure 13.

This right hand side is identical to an  $[L_1, N_1, N_2]$ -configuration, apart from the fact that it includes an  $(N_3 \times N_3)$ -domain wall configuration, that corresponds to the spin-down reference state contribution of  ${}_r\langle N_3^\vee|$ , that is not included in an  $[L_1, N_1, N_2]$ -configuration.

**Step 2. The domain wall partition functions.** Accounting for the domain wall partition function, and working in the homogeneous limit where all quantum rapidities are set to  $z = \frac{1}{2}\sqrt{-1}$ , we obtain our result for the structure constants, which up to a factor, is in determinant form.

$$(68) \quad c_{123}^{(0)} = \mathcal{N}_{123} Z_{N_3}^{hom} \left( \{w\}_{N_3}, \frac{1}{2}\sqrt{-1} \right) S^{hom}[L_1, N_1, N_2] \left( \{u_\beta\}_{N_1}, \{v\}_{N_2}, \frac{1}{2}\sqrt{-1} \right)$$

where the normalization  $\mathcal{N}_{123}$  is defined in (64), the  $(N_3 \times N_3)$  domain wall partition function  $Z_{N_3}^{hom} \left( \{w\}_{N_3}, \frac{1}{2}\sqrt{-1} \right)$  is given in (54). The term  $S^{hom}[L_1, N_1, N_2] \left( \{u_\beta\}_{N_1}, \{v\}_{N_2}, \frac{1}{2}\sqrt{-1} \right)$  is an  $(N_1 \times N_1)$  determinant expression of the partition function of an  $[L_1, N_1, N_2]$ -configuration,



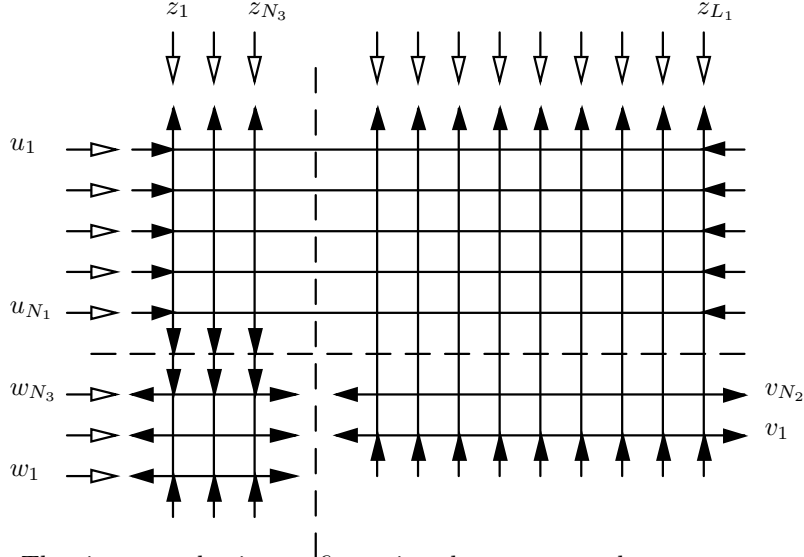


Figure 14: The six-vertex lattice configuration that corresponds, up to a normalization factor  $\mathcal{N}_{123}$ , to the structure constant  $c_{123}^{(0)}$ .

given in (47). The auxiliary rapidities  $\{u\}$ ,  $\{v\}$  and  $\{w\}$  are those of the eigenstates  $\mathcal{O}_1$ ,  $\mathcal{O}_2$  and  $\mathcal{O}_3$  in [20], respectively. Notice that  $\{v\}$  and  $\{w\}$  are actually  $\{v\}_\beta$  and  $\{w\}_\beta$ , that is, they satisfy Bethe equations, but this fact is not used.

**4.10. Type-A specializations.** Equation 68 is quite general. To obtain an expression specific to a certain Type-A theory, we need to use the values of the spin-chain parameters appropriate to that theory, as were given in Subsection 0.7. All Type-A theories map to XXX spin- $\frac{1}{2}$  chains, hence the anisotropy parameter  $\Delta = 1$ , but with different values for the twist parameter  $\theta$ . Theory 1 is  $\text{SYM}_4$  and  $\theta = 0$ . Theory 2 is  $\text{SYM}_4^M$  is an Abelian orbifold version of  $\text{SYM}_4$  and  $\theta = \frac{2\pi}{M}$ . Theory 3 is a real- $\beta$ -deformed version of  $\text{SYM}_4$  and  $\theta = \beta$ .

## 5. STRUCTURE CONSTANTS IN TYPE-B THEORIES

In this section, we consider structure constants in Type-B theories. Our approach is parallel to that used in Type-A. The difference is that each Type-B theory has only one doublet, and therefore requires a slightly modified treatment<sup>18</sup>.

In type-A theories, the left part of  $\mathcal{O}_2$  gets trivially contracted with the right part of  $\mathcal{O}_3$ , and the pants diagram is reduced to the ‘*narrow pants*’ diagram in Figure 13. As we will see, the starting point in the case of Type-B theories is a ‘*narrow pants*’ diagram.

This implies that in Type-B theories  $\mathcal{O}_3$  must be chosen to be a BPS-like state, with *one* type of fundamental field in the composite operator  $\mathcal{O}_3$ . On the other hand, since the missing contraction (that between the left part of  $\mathcal{O}_2$  and the right part of  $\mathcal{O}_3$ ) was trivial for Type-A theories, the final result remains the same.

**5.1. Type-B. Fundamental field content of the states.** As in Type-A, we consider single-trace operators in an  $SU(2)$  sector of a 1-loop conformally-invariant gauge theory, that is  $\text{Tr}(f_1 f_2 f_3 \dots)$ , where  $f_i \in \{u, d\}$  is a fundamental field that belongs to an  $SU(2)$  doublet.

The new feature in Type-B theories is that we have only one doublet to work with. The doublets relevant to Type-B theories were given in Subsection 0.7. Theory 4 is pure gauge  $\text{SYM}_2$ , and the doublet consists of the gluino and its conjugate  $\{\lambda, \bar{\lambda}\}$ . Theory 5 is pure gauge  $\text{SYM}_1$ , and the doublet consists of the complex scalar and its conjugate  $\{\phi, \bar{\phi}\}$ . Theory 6 is

<sup>18</sup>The conclusion that, in order to obtain a determinant formula, one of the single-trace operators should be BPS-like, was obtained in discussions with C Ahn and R Nepomechie.

pure QCD and the doublet consists of the light cone derivative of the gauge field component  $A$  and its conjugate  $\bar{A}$ , that is,  $\{\partial_+ A, \partial_+ \bar{A}\}$ . In the following, we deal with all three theories in one go, using the notation  $\{\zeta, \bar{\zeta}\}$  for a generic single doublet.

Since we have only one doublet to construct composite operators from, we identify the fundamental field content of  $\mathcal{O}_i$ ,  $i \in \{1, 2, 3\}$  with spin-chain spin states as shown in Table 2.

Operator	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \end{pmatrix}$
$\mathcal{O}_1$	$\zeta$	$\bar{\zeta}$	$\bar{\zeta}$	$\zeta$
$\mathcal{O}_2$	$\bar{\zeta}$	$\zeta$	$\zeta$	$\bar{\zeta}$
$\mathcal{O}_3$	$\bar{\zeta}$	$\zeta$	$\zeta$	$\bar{\zeta}$

TABLE 2. Identification of Type-**B** operator content of  $\mathcal{O}_i$ ,  $i \in \{1, 2, 3\}$  with initial and final spin-chain states.

Once again, in our conventions

$$(69) \quad \langle \bar{\zeta} | \zeta \rangle = \langle \zeta | \zeta \rangle = 1, \quad \langle \zeta | \zeta \rangle = \langle \bar{\zeta} | \bar{\zeta} \rangle = 0$$

From Table 2, one can read the fundamental-scalar operator content of each single-trace operator  $\mathcal{O}_i$ ,  $i \in \{1, 2, 3\}$ , when it is an initial state and when it is a final state.

**5.2. Similarities between Type-A and Type-B theories.** Steps 1, 2 and 3 from the EGSV construction of the structure constants apply unchanged to Type-**B** theories. In other words, **1.** The splitting of each lattice, **2.** The flipping procedure, and **3.** The contraction of left and right halves to form scalar products, are replicated in the case of Type-**B** theories. Therefore we see that equation (63) continues to hold, and we assume that as our starting point.

**5.3. Differences between Type-A and Type-B theories.** **1.** In the case of Type-**A** theories,  $\mathcal{O}_3$  contains  $Z$  fields that can only contract with  $\bar{Z}$  fields in  $\mathcal{O}_2$ . This is because there are no fields that they can contract with in  $\mathcal{O}_1$ . This trivializes the  ${}_l \langle \mathcal{O}_2 | \mathcal{O}_3 \rangle_r$  scalar product.

This is not the case in Type-**B** theories, where we have only a single doublet that must be used to populate all three states  $\mathcal{O}_1$ ,  $\mathcal{O}_2$  and  $\mathcal{O}_3$ . Because of that, one can see that if there is a contraction between  $\mathcal{O}_2$  and  $\mathcal{O}_3$ , it is in general non-trivial. This is sufficient to prevent us from duplicating our Type-**A** arguments in the case of Type-**B** theories. In fact, there is yet another difference.

**2.** In the case of Type-**A** theories,  $\mathcal{O}_3$  contains  $\bar{X}$  fields that can contract only with  $X$  fields in  $\mathcal{O}_1$ . The reason is that there are no  $X$  fields in  $\mathcal{O}_2$ . This trivializes the scalar product that involves the left part of  $\mathcal{O}_1$  and the right part of  $\mathcal{O}_3$ , leading to a domain wall partition function.

Once again, in the case of Type-**B** theories, the above trivial contraction is no longer the case, and contractions between  $\mathcal{O}_1$  and  $\mathcal{O}_3$  are in general non-trivial.

**5.4. One of the operators must be BPS-like.** Because of the above reasons, *we cannot map the most general  $SU(2)$  structure constants of Type-**B** operators onto a restricted Slavnov scalar product.* However, both problems are overcome if we take  $\mathcal{O}_3$  to be BPS-like, that is, a single-trace operator of the form  $\text{Tr} [\bar{\zeta} \bar{\zeta} \cdots \bar{\zeta}]$ . This means that we *demand* that  $N_3 = L_3$ , or equivalently, that  $l_{23} = L_3 - N_3 = 0$ . In other words, the fields in  $\mathcal{O}_3$  are all of the same type  $\bar{\zeta}$  (magnons) and they contract with a subset of the fields in  $\mathcal{O}_1$ , while there are no contractions between  $\mathcal{O}_3$  and  $\mathcal{O}_2$ . From this, we conclude that the starting point of the Type-**B** structure constants that we can compute in determinant form is the ‘narrow pants’ diagram in Figure 13.

But we know that the partition function of the lattice configuration corresponding to Figure 13 is given by a restricted Slavnov scalar product. Therefore for Type-**B** structure constants for which  $\mathcal{O}_3$  is BPS-like, that is  $L_3 = N_3$ , we obtain

$$(70) \quad c_{123}^{(0)} = \mathcal{N}_{123} Z_{N_3}^{hom} \left( \{w\}_{N_3}, \frac{1}{2}\sqrt{-1} \right) S^{hom}[L_1, N_1, N_2] \left( \{u_\beta\}_{N_1}, \{v\}_{N_2}, \frac{1}{2}\sqrt{-1} \right)$$

This is the same result as the Type-**A** case, but with the caveat that we are restricting our attention to the situation  $L_3 = N_3$ . As a result the Gaudin norm  $\mathcal{N}_3$ , which occurs in the normalization factor  $\mathcal{N}_{123}$ , is equal to the partition function of a  $BC$ -configuration with length- $N_3$  initial and final reference states, and  $N_3$   $B$ -lines and  $C$ -lines. As we commented in Subsection 3.17, such a configuration factorizes into a product of domain wall partition functions. Hence we are able to cancel the factor  $Z_{N_3}^{hom} \left( \{w\}_{N_3}, \frac{1}{2}\sqrt{-1} \right)$  in (70) at the expense of the factor  $\sqrt{\mathcal{N}_3}$  in the denominator, and obtain the final expression

$$(71) \quad c_{123}^{(0)} = \sqrt{\frac{L_1 L_2 L_3}{\mathcal{N}_1 \mathcal{N}_2}} S^{hom}[L_1, N_1, N_2] \left( \{u_\beta\}_{N_1}, \{v\}_{N_2}, \frac{1}{2}\sqrt{-1} \right)$$

**5.5. Type-**B** specializations.** As in the previous section, (70) is quite general. To obtain an expression specific to a certain Type-**B** theory, we need to use the values of the spin-chain parameters appropriate to that theory, as were given in Subsection 0.7. All Type-**B** theories map to periodic XXZ spin- $\frac{1}{2}$  chains, hence the twist parameter  $\theta = 0$ , but with different values of the anisotropy parameter  $\Delta$ . Theory 4 is pure  $\text{SYM}_2$  and  $\Delta = 3$  [28, 7]. Theory 5 is pure  $\text{SYM}_1$  and  $\Delta = \frac{1}{2}$  [7]. Theory 6 is pure gauge QCD and  $\Delta = -\frac{11}{3}$  [7].

## 6. DISCRETE KP $\tau$ -FUNCTIONS

In this section we closely follow [37], where it was shown that Slavnov's scalar product is a  $\tau$ -function of the discrete KP hierarchy. The only differences in this work are **1.** A more compact expression for the  $\tau$ -function itself, see (100), **2.** The inclusion of the twist parameter  $\theta$  in the  $\tau$ -function, and **3.** A discussion of restricting the Miwa variables to the values of the quantum inhomogeneities.

**6.1. Notation related to sets of variables.** We use  $\{x\}$  for the set of finitely many variables  $\{x_1, x_2, \dots, x_N\}$ , and  $\{\hat{x}_m\}$  for  $\{x\}$  with the element  $x_m$  omitted. In the case of sets with a repeated variable  $x_i$ , we use the superscript  $(m_i)$  to indicate the multiplicity of  $x_i$ , as in  $x_i^{(m_i)}$ . For example,  $\{x_1^{(3)}, x_2, x_3^{(2)}, x_4, \dots\}$  is the same as  $\{x_1, x_1, x_1, x_2, x_3, x_3, x_4, \dots\}$  and  $f\{\dots, x_i^{(m_i)}, \dots\}$  is equivalent to saying that  $f$  depends on  $m_i$  distinct variables all of which have the same value  $x_i$ . For simplicity, we use  $x_i$  to indicate  $x_i^{(1)}$ .

**6.2. The complete symmetric function  $h_i\{x\}$ .** Let  $\{x\}$  denote a set of  $N$  variables  $\{x_1, x_2, \dots, x_N\}$ . The complete symmetric function  $h_i\{x\}$  is the coefficient of  $k^i$  in the power series expansion

$$(72) \quad \prod_{i=1}^N \frac{1}{1 - x_i k} = \sum_{i=0}^{\infty} h_i\{x\} k^i$$

For example,  $h_0\{x\} = 1$ ,  $h_1(x_1, x_2, x_3) = x_1 + x_2 + x_3$ ,  $h_2(x_1, x_2) = x_1^2 + x_1 x_2 + x_2^2$ , and  $h_i\{x\} = 0$  for  $i < 0$ .

**6.3. Useful identities for  $h_i\{x\}$ .** From (72), it is straightforward to show that

$$(73) \quad h_i\{x\} = h_i\{\hat{x}_m\} + x_m h_{i-1}\{x\}$$

Then from (73) one obtains

$$(74) \quad (x_m - x_n) h_{i-1}\{x\} = h_i\{\hat{x}_n\} - h_i\{\hat{x}_m\}$$

$$(75) \quad (x_m - x_n)h_i\{x\} = x_m h_i\{\widehat{x}_n\} - x_n h_i\{\widehat{x}_m\}$$

**6.4. Discrete derivatives.** The discrete derivative  $\Delta_m h_i\{x\}$  of  $h_i\{x\}$  with respect to any one variable  $x_m \in \{x\}$  is defined using (73) as

$$(76) \quad \Delta_m h_i\{x\} = \frac{h_i\{x\} - h_i\{\widehat{x}_m\}}{x_m} = h_{i-1}\{x\}$$

Note that the effect of applying  $\Delta_m$  to  $h_i\{x\}$  is a complete symmetric function  $h_{i-1}\{x\}$  of degree  $i - 1$  in the same set of variables  $\{x\}$ .

**6.5. The discrete KP hierarchy.** Discrete KP is an infinite hierarchy of integrable partial difference equations in an infinite set of continuous Miwa variables  $\{x\}$ , where time evolution is obtained by changing the multiplicities  $\{m\}$  of these variables. In this work, we are interested in the situation where the total number of continuous Miwa variables is finite, which corresponds to setting to zero all continuous Miwa variables apart from  $\{x_1, \dots, x_N\}$ . In this case, the discrete KP hierarchy can be written in bilinear form as the  $n \times n$  determinant equations

$$(77) \quad \det \begin{pmatrix} 1 & x_1 & \cdots & x_1^{n-2} & x_1^{n-2}\tau_{+1}\{x\}\tau_{-1}\{x\} \\ 1 & x_2 & \cdots & x_2^{n-2} & x_2^{n-2}\tau_{+2}\{x\}\tau_{-2}\{x\} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & \cdots & x_n^{n-2} & x_n^{n-2}\tau_{+n}\{x\}\tau_{-n}\{x\} \end{pmatrix} = 0$$

where  $3 \leq n \leq N$ , and

$$(78) \quad \begin{aligned} \tau_{+i}\{x\} &= \tau\{x_1^{(m_1)}, \dots, x_i^{(m_i+1)}, \dots, x_N^{(m_N)}\} \\ \tau_{-i}\{x\} &= \tau\{x_1^{(m_1+1)}, \dots, x_i^{(m_i)}, \dots, x_N^{(m_N+1)}\} \end{aligned}$$

In other words, if  $\tau\{x\}$  has  $m_i$  copies of the variable  $x_i$ , then  $\tau_{+i}\{x\}$  has  $m_i + 1$  copies of  $x_i$  and the multiplicities of all other variables remain the same, while  $\tau_{-i}\{x\}$  has one more copy of each variable except  $x_i$ . Equivalently, one can use the simpler notation

$$(79) \quad \begin{aligned} \tau_{+i}\{x\} &= \tau\{m_1, \dots, (m_i + 1), \dots, m_N\} \\ \tau_{-i}\{x\} &= \tau\{(m_1 + 1), \dots, m_i, \dots, (m_N + 1)\} \end{aligned}$$

The simplest discrete KP bilinear difference equation, in the notation of (79), is

$$(80) \quad \begin{aligned} &x_i(x_j - x_k)\tau\{m_i + 1, m_j, m_k\}\tau\{m_i, m_j + 1, m_k + 1\} \\ &+ x_j(x_k - x_i)\tau\{m_i, m_j + 1, m_k\}\tau\{m_i + 1, m_j, m_k + 1\} \\ &+ x_k(x_i - x_j)\tau\{m_i, m_j, m_k + 1\}\tau\{m_i + 1, m_j + 1, m_k\} = 0 \end{aligned}$$

where  $\{x_i, x_j, x_k\} \in \{x\}$  and  $\{m_i, m_j, m_k\} \in \{m\}$  are any two (corresponding) triples in the sets of continuous and discrete (integral valued) Miwa variables. Equation (80) is the discrete analogue of the KP equation in continuous time variables.

**6.6. Casoratian matrices and determinants.** A Casoratian matrix  $\Omega$  of the type that appears in this paper is such that its matrix elements  $\omega_{ij}$  satisfy

$$(81) \quad \omega_{i,j+1}\{x\} = \Delta_m \omega_{ij}\{x\}$$

where the discrete derivative  $\Delta_m$  is taken with respect to any one variable  $x_m \in \{x\}$  (it is redundant to specify which variable, since  $\omega_{ij}\{x\}$  is symmetric in  $\{x\}$ ). From the definition of the discrete derivative  $\Delta_m$ , it is clear that the entries of Casoratian matrices satisfy

$$(82) \quad \omega_{ij}\{x_1, \dots, x_m^{(2)}, \dots, x_N\} = \omega_{ij}\{x_1, \dots, x_N\} + x_m \omega_{i,j+1}\{x_1, \dots, x_m^{(2)}, \dots, x_N\}$$

which, in turn, gives rise to the identity

$$(83) \quad (x_r - x_s) \omega_{ij} \{x_1, \dots, x_r^{(2)}, \dots, x_s^{(2)}, \dots, x_N\} = x_r \omega_{ij} \{x_1, \dots, x_r^{(2)}, \dots, x_N\} - x_s \omega_{ij} \{x_1, \dots, x_s^{(2)}, \dots, x_N\}$$

If  $\Omega$  is a Casoratian matrix, then  $\det \Omega$  is a Casoratian determinant. Casoratian determinants are discrete analogues of Wronskian determinants.

**6.7. Notation for column vectors with elements  $\omega_{ij}$ .** We need the column vector

$$(84) \quad \vec{\omega}_j = \begin{pmatrix} \omega_{1j} \{x_1^{(m_1)}, \dots, x_N^{(m_N)}\} \\ \omega_{2j} \{x_1^{(m_1)}, \dots, x_N^{(m_N)}\} \\ \vdots \\ \omega_{Nj} \{x_1^{(m_1)}, \dots, x_N^{(m_N)}\} \end{pmatrix}$$

and write

$$(85) \quad \vec{\omega}_j^{[k_1, \dots, k_n]} = \begin{pmatrix} \omega_{1j} \{x_1^{(m_1)}, \dots, x_{k_1}^{(m_{k_1}+1)}, \dots, x_{k_n}^{(m_{k_n}+1)}, \dots, x_N^{(m_N)}\} \\ \omega_{2j} \{x_1^{(m_1)}, \dots, x_{k_1}^{(m_{k_1}+1)}, \dots, x_{k_n}^{(m_{k_n}+1)}, \dots, x_N^{(m_N)}\} \\ \vdots \\ \omega_{Nj} \{x_1^{(m_1)}, \dots, x_{k_1}^{(m_{k_1}+1)}, \dots, x_{k_n}^{(m_{k_n}+1)}, \dots, x_N^{(m_N)}\} \end{pmatrix}$$

for the corresponding column vector where the multiplicities of the variables  $x_{k_1}, \dots, x_{k_n}$  are increased by 1.

**6.8. Notation for determinants with elements  $\omega_{ij}$ .** We also need the determinant

$$(86) \quad \tau = \det \begin{pmatrix} \vec{\omega}_1 & \vec{\omega}_2 & \dots & \vec{\omega}_N \end{pmatrix} = \begin{vmatrix} \vec{\omega}_1 & \vec{\omega}_2 & \dots & \vec{\omega}_N \end{vmatrix}$$

and the notation

$$(87) \quad \tau^{[k_1, \dots, k_n]} = \begin{vmatrix} \vec{\omega}_1^{[k_1, \dots, k_n]} & \vec{\omega}_2^{[k_1, \dots, k_n]} & \dots & \vec{\omega}_N^{[k_1, \dots, k_n]} \end{vmatrix}$$

for the determinant with shifted multiplicities.

**6.9. Identities satisfied by Casoratian determinants.** Two identities, which are needed in the sequel, are

$$(88) \quad x_1^{n-2} \tau^{[1]} = \begin{vmatrix} \vec{\omega}_1 & \vec{\omega}_2 & \dots & \vec{\omega}_{N-1} & \vec{\omega}_{N-n+2}^{[1]} \end{vmatrix}$$

$$(89) \quad \prod_{1 \leq r < s \leq n} (x_r - x_s) \tau^{[1, \dots, n]} = \begin{vmatrix} \vec{\omega}_1 & \dots & \vec{\omega}_{N-n} & \vec{\omega}_{N-n+1}^{[n]} & \vec{\omega}_{N-n+1}^{[n-1]} & \dots & \vec{\omega}_{N-n+1}^{[1]} \end{vmatrix}$$

These identities may be proved by using the (82) and (83) to perform column operations in the determinant expressions for  $\tau^{[1]}$  and  $\tau^{[1, \dots, n]}$ . To keep the exposition concise we do not present these proofs, but full details can be found in [37].

**6.10. Casoratians are discrete KP  $\tau$ -functions.** Following [38], consider the  $2N \times 2N$  determinant

$$(90) \quad \det \begin{pmatrix} \vec{\omega}_1 & \dots & \vec{\omega}_{N-1} & \vec{\omega}_{N-n+2}^{[1]} & 0_1 & \dots & 0_{N-n+1} & \vec{\omega}_{N-n+2}^{[n]} \dots \vec{\omega}_{N-n+2}^{[2]} \\ 0_1 & \dots & 0_{N-1} & \vec{\omega}_{N-n+2}^{[1]} & \vec{\omega}_1 & \dots & \vec{\omega}_{N-n+1} & \vec{\omega}_{N-n+2}^{[n]} \dots \vec{\omega}_{N-n+2}^{[2]} \end{pmatrix} = 0$$

which is identically zero. For notational clarity, we have used subscripts to label the position of columns of zeros. Performing a Laplace expansion of the left hand side of (90) in  $N \times N$  minors along the top  $N \times 2N$  block, we obtain

$$(91) \quad \sum_{k=1}^n (-)^{k-1} |\vec{\omega}_1 \cdots \vec{\omega}_{N-1} \vec{\omega}_{N-n+2}^{[k]}| \times \\ |\vec{\omega}_1 \cdots \vec{\omega}_{N-n+1} \vec{\omega}_{N-n+2}^{[n]} \cdots \vec{\omega}_{N-n+2}^{[k+1]} \vec{\omega}_{N-n+2}^{[k-1]} \cdots \vec{\omega}_{N-n+2}^{[1]}| = 0$$

By virtue of (88) and (89), (91) becomes

$$(92) \quad \sum_{k=1}^n (-)^{k-1} x_k^{n-2} \tau^{[k]} \prod_{\substack{1 \leq r < s \leq n \\ r, s \neq k}} (x_r - x_s) \tau^{[1, \dots, \hat{k}, \dots, n]} = 0$$

Using the Vandermonde determinant identity

$$(93) \quad \det \begin{pmatrix} 1 & x_1 & \cdots & x_1^{n-2} \\ \vdots & \vdots & & \vdots \\ \langle 1 & x_k & \cdots & x_k^{n-2} \rangle \\ \vdots & \vdots & & \vdots \\ 1 & x_n & \cdots & x_n^{n-2} \end{pmatrix} = \prod_{\substack{1 \leq r < s \leq n \\ r, s \neq k}} (x_r - x_s)$$

with  $\langle 1 \ x_k \ \dots \ x_k^{n-2} \rangle$  denoting the omission of the  $k$ -th row of the matrix, we recognize (92) as the cofactor expansion of the determinant in (77) along its last column. Hence we conclude that Casoratian determinants satisfy the bilinear difference equations of discrete KP.

**6.11. Change of variables.** To interpret the Slavnov determinant (45) as a  $\tau$ -function of discrete KP in the sense described above, it is necessary to adopt a change of variables as follows

$$(94) \quad \{e^{-2v_i}, e^{2u_{\beta_i}}, e^{2z_i}, e^{2\eta}\} \rightarrow \{x_i, y_i, z_i, q\}$$

In other words, our new variables (of which  $\{x_1, \dots, x_N\}$  end up being the continuous Miwa variables of discrete KP) are expressed as exponentials of the original variables. Furthermore, we consider a new normalization of the scalar product, given by

$$(95) \quad \mathbb{S}[L, N, N] = e^{N^2 \eta} \prod_{i=1}^N e^{(L-1)(u_{\beta_i} - v_i)} \prod_{i=1}^L e^{2N z_i} \prod_{j=1}^N \prod_{k=1}^L [v_j - z_k][u_{\beta_j} - z_k] S[L, N, N]$$

Applying this normalization to (45), performing trivial rearrangements within the determinant and making the change of variables as prescribed by (94), we obtain

$$(96) \quad \mathbb{S}[L, N, N] = \frac{(q-1)^N \prod_{i=1}^N \prod_{j=1}^L (y_i - z_j)}{\prod_{1 \leq i < j \leq N} (x_i - x_j)(y_i - y_j)} \times \\ \det \left( \frac{e^{-2i\theta} q^{N-1} \prod_{k \neq i}^N \left(1 - x_j \frac{y_k}{q}\right) \prod_{k=1}^L (1 - x_j z_k)}{1 - x_j y_i} - \frac{q^{\frac{L}{2}} \prod_{k \neq i}^N (1 - q x_j y_k) \prod_{k=1}^L \left(1 - x_j \frac{z_k}{q}\right)}{1 - x_j y_i} \right)_{1 \leq i, j \leq N}$$

Our goal is to show that  $\mathbb{S}[L, N, N]$  has the form of a Casoratian determinant, where the discrete derivative is taken with respect to the variables  $\{x_1, \dots, x_N\}$ .

**6.12. Removing the pole in the Slavnov scalar product.** For all  $1 \leq i \leq N$ , define the function  $\gamma_i$  as

$$(97) \quad \gamma_i = e^{-2i\theta} q^{N-1} \prod_{j \neq i}^N \left(1 - \frac{y_j}{qy_i}\right) \prod_{j=1}^L \left(1 - \frac{z_j}{y_i}\right) - q^{\frac{L}{2}} \prod_{j \neq i}^N \left(1 - \frac{qy_j}{y_i}\right) \prod_{j=1}^L \left(1 - \frac{z_j}{qy_i}\right)$$

These functions provide a convenient way of expressing the Bethe equations (35) under the change of variables (94), namely

$$(98) \quad \gamma_i = 0, \quad \text{for all } 1 \leq i \leq N.$$

Recalling that these equations are assumed to apply to the variables  $\{y_1, \dots, y_N\}$ , we see that the pole at  $x_j = 1/y_i$  in the determinant of (96) can be removed. We omit the details here as they are mechanical, and state only the result of this calculation, which reads

$$(99) \quad \mathbb{S}[L, N, N] = \frac{(q-1)^N \prod_{i=1}^N \prod_{j=1}^L (y_i - z_j)}{\prod_{1 \leq i < j \leq N} (x_i - x_j)(y_i - y_j)} \det \left( \sum_{k=0}^{L+N-2} \left[ y_i^k \gamma_i \right]_+ x_j^k \right)_{1 \leq i, j \leq N}$$

where  $[y_i^k \gamma_i]_+$  denotes all terms in the Laurent expansion of  $y_i^k \gamma_i$  which have non-negative degree in  $y_i$ .

**6.13. The Slavnov scalar product is a discrete KP  $\tau$ -function.** Using identities (73) and (74) to perform elementary column operations in the determinant of (99), it is possible to remove the Vandermonde  $\prod_{1 \leq i < j \leq N} (x_i - x_j)$  from the denominator of this equation. This procedure is directly analogous to the proof of the Jacobi-Trudi identity for Schur functions [39]. The result obtained is

$$(100) \quad \mathbb{S}[L, N, N] = \frac{(q-1)^N \prod_{i=1}^N \prod_{j=1}^L (y_i - z_j)}{\prod_{1 \leq i < j \leq N} (y_j - y_i)} \det \left( \sum_{k=0}^{L+N-2} \left[ y_i^k \gamma_i \right]_+ h_{k-j+1} \{x\} \right)_{1 \leq i, j \leq N}$$

Up to an overall multiplicative factor which does not depend on the variables  $\{x\}$ , the normalized scalar product  $\mathbb{S}[L, N, N]$  is a determinant of the form  $\det \Omega$ , where the matrix  $\Omega$  has entries  $\omega_{ij}$  which satisfy

$$(101) \quad \omega_{i,j+1} = \Delta_m \omega_{i,j}, \quad \omega_{i,1} = \sum_{k=0}^{L+N-2} \left[ y_i^k \gamma_i \right]_+ h_k \{x\}$$

Hence  $\mathbb{S}[L, N, N]$  has the form of a Casoratian determinant, making it a discrete KP  $\tau$ -function in the variables  $\{x\} = \{x_1, \dots, x_N\}$ .

**6.14. Restrictions of  $\mathbb{S}[L, N_1, N_1]$ .** Similarly to (95), we define a new normalization of the restricted scalar product  $S[L, N_1, N_2]$  as follows

$$(102) \quad \mathbb{S}[L, N_1, N_2] = e^{N_1^2 \eta} \prod_{i=1}^{N_1} e^{(L-1)u_{\beta_i}} \prod_{i=1}^{N_2} e^{-(L-1)v_i} \prod_{i=1}^L e^{(N_1+N_2)z_i} \\ \times \prod_{j=1}^{N_2} \prod_{k=1}^L [v_j - z_k] \prod_{j=1}^{N_1} \prod_{k=1}^L [u_{\beta_j} - z_k] S[L, N_1, N_2]$$

Normalizing both sides of (46) using (95) and (102), and working in terms of the variables introduced by (94), we obtain the result

$$(103) \quad \mathbb{S}[L, N_1, N_1] \Big|_{\substack{x_{N_1}=1/z_1 \\ \vdots \\ x_{(N_2+1)}=1/z_{N_3}}} = (z_1 \dots z_{N_3})^{1/2} \prod_{i=1}^{N_3} \prod_{j=1}^L (q^{1/2} - q^{-1/2} z_j / z_i) \mathbb{S}[L, N_1, N_2]$$

Hence the function  $\mathbb{S}[L, N_1, N_2]$  is (up to an overall multiplicative factor) a restriction of  $\mathbb{S}[L, N_1, N_1]$ , obtained by setting the variables  $x_{N_1}, \dots, x_{N_2+1}$  to the values  $1/z_1, \dots, 1/z_{N_3}$ . Since  $\mathbb{S}[L, N_1, N_1]$  is a discrete KP  $\tau$ -function in the variables  $\{x_1, \dots, x_{N_1}\}$ , it is clear that  $\mathbb{S}[L, N_1, N_2]$  is also a  $\tau$ -function in the unrestricted set of variables  $\{x_1, \dots, x_{N_2}\}$ .

## 7. SUMMARY AND COMMENTS

Following [21], we obtained determinant expressions for two types of structure constants.

**1.** structure constants of non-extremal 3-point functions of single-trace non-BPS operators in the scalar sector of  $\text{SYM}_4$  and two close variations on it (an Abelian orbifolding of  $\text{SYM}_4$  and a real- $\beta$ -deformation of it). The operators involved map to states in closed XXX spin- $\frac{1}{2}$  chains, that are periodic in the case of  $\text{SYM}_4$ , and twisted in the other two cases. **2.** structure constants of extremal 3-point functions of two non-BPS and one BPS single-trace operators in (not necessarily scalar, but spin-zero) sectors of pure gauge  $\text{SYM}_2$ ,  $\text{SYM}_1$  and QCD. The operators involved map to states in closed periodic XXZ spin- $\frac{1}{2}$  chains, with different values of the anisotropy parameter, as identified in [28, 7]. One of the operators must be BPS-like.

Our expressions are basically special cases of Slavnov's determinant for the scalar product of a Bethe eigenstate and a generic state in a (generally twisted) closed XXZ spin chain. Finally, following [37], we showed that all these determinants are discrete KP  $\tau$ -functions, in the sense that they obey the Hirota-Miwa equations.

The study of 3-point functions is a continuing activity. In [42], a systematic study, using perturbation theory, of 3-point functions in planar  $\text{SYM}_4$  at 1-loop level, involving scalar field operators up to length 5 is reported on. In [43], quantum corrections to 3-point functions of the very same type studied in this work planar  $\text{SYM}_4$  are studied using integrability. At 1-loop level, new algebraic structures are found that govern all 2-loop corrections to the mixing of the operators as well as automatically incorporate all 1-loop corrections to the tree-level computations.

In [44], operator product expansions of local single-trace operators composed of self-dual components of the field strength tensor in planar QCD are considered. Using methods that extend those used in this work to spin-1 chains, a determinant expression for certain tree-level structure constants that appear in the operator product expansion is obtained. More recently, in [45], the classical limit of the determinant form of the structure constants that appear in this work, was obtained.

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